2020 ARIZONA WINTER SCHOOL COURSE AND PROJECT OUTLINE: "ABELIAN CHABAUTY"

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1. Course outline

Let X be a "nice"[1] curve over a number field K. This course will discuss the method of Chabauty and Coleman (alternatively, "Abelian Chabauty"), which is a p-adic technique that often allows one to provably compute the (finite, by Faltings' theorem) set $X(K)$ of K-rational points on X, and which is an essential part of the "explicit approaches to rational points" toolbox.

Abelian Chabauty. We will start with a detailed discussion of the method of Chabauty and Coleman, and will address every point of view (theoretical, practical, computational). We will include some discussion of how abelian Chabauty is a special case of non-abelian Chabauty. Poonen will address some subset of this material in his plenary lecture.

Uniform Bounds. The *uniformity conjecture* is one of the outstanding open conjectures in arithmetic and diophantine geometry, and asserts that, for a number field K, there exists a constant $B(g, K)$ such that every smooth curve X over K of genus $g \geq 2$ has at most $B(g, K)$ many K-rational points. The uniformity conjecture famously follows from the conjecture of Lang–Vojta (the higher dimension analogue of Faltings' theorem). We will discuss techniques for using p-adic methods to obtain uniform bounds on small rank curves, including Coleman's "Effective Chabuaty" [Col85].

Bad reduction. One avenue to improve on Coleman's bound is to generalize the Chabauty framework and Coleman's arguments to the case of bad reduction. We will discuss the advantage of working at bad primes and the difficulties that arise, starting with the work of Lorenzini–Tucker [LT02].

Rank favorable bounds. Stoll [Sto06] was the first to discover "rank favorable" bounds: when the rank is strictly smaller than $g-1$, there are more "inputs" to Chabauty's method and one expects this extra flexibility to lead to improvements in the method. The "geometric input" that allows Stoll to convert this intuition into a theorem is the (classical) notion of "rank of a divisor", and after translating his setup into this language, improved bounds follow from Clifford's Theorem. I'll discuss my work with Eric Katz [KZB12] which generalizes Stoll's theorem to the bad reduction setting, and which exploits recent ideas from tropical geometry (in particular "chip-firing", the "discrete case" of tropical geometry).

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[1] i.e., smooth projective geometrically integral

Tropical Geometry. For a curve with bad reduction at a prime p, it had been well understood that "monodromy" and "analytic continuation" of p-adic integrals was an issue. Coleman proved that in the case of good reduction, there is no "monodromy" and the various ways of analytically continuing p-adic integrals all coincide. In the case of bad reduction, they generally do not coincide (we will discuss a simple example which illustrates this).

Stoll [Sto19] discovered that, while choices of analytic continuation genuinely do differ, they do so in a fairly controlled manner (linear, even), and was able to exploit this to prove a uniformity result for *hyperelliptic* curves of small rank.

These results all argue in the framework of rigid geometry (in the sense of Tate). Enter Berkovich spaces, which fill in the "missing" points of rigid spaces and which, at least in the case of curves, are fairly concrete and manageable topological spaces (they're even Hausdorff). I'll discuss Chabauty in the setting of Berkovich and tropical geometry and explain how modern tools (e.g., Berkovich's contraction theorem and Thuillier's slope formula, exposited in [BPR13]) give a clean explanation of Coleman's "good reduction" theorem, and will discuss my work with Katz and Rabinoff [KRZB] which give uniform bounds for arbitrary (but still small rank) curves.

Background reading. I recommend McCallum and Poonen's survey [MP12] as a great starting point for the method of Chabauty and Coleman, and my survey with Katz and Rabinoff [KRZB16] for tropical techniques.

2. Projects

(1) Find every rational point on every (symmetric power of every) modular curve. More seriously: there are several interesting example of modular (in some appropriate sense) curves, and one collection of projects is to study, via Chabauty and other explicit methods, the rational points on these curves.

As an example: there are several composite, but non prime power, level modular curves that arise in naturally in "Mazur's program B" for which the determination of rational points has some particular challenging aspect.

(2) Similarly, one could compute specific quadratic points on certain modular curves using a modified Chabauty. More precisely, there is a heuristic of Siksek and Wetherell saying that for a nice curve $X/\mathbb{Q}$, a Chabauty-type method could bound the number of K-rational points on a curve X of genus g under the weaker assumption that $J_X(K)$ has rank $r \leq d(g-1)$ where $d = [K : \mathbb{Q}]$.

(3) Improve the "rank favorable" bounds on the rank functions that arise in Stoll's work and in my work with Eric Katz for special curves (e.g., trigonal). This project would involve very little p-adic analysis; the techniques are more akin to the geometry of curves and combinatorics.

(4) Uniform bounds for dth symmetric products of curves with small rank.

(5) Rank favorable, uniform bounds for projective plane curves (e.g., non-hyperelliptic genus 3 curves) with small rank. For many "special" families of curves one has an explicit description of differentials, which helps with the "p-adic analysis" part of the arguments.

References

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[Sto06] Michael Stoll, *Independence of rational points on twists of a given curve*, Compos. Math. **142** (2006), no. 5, 1201–1214. MR2264661 (2007m:14025) ↑1

[Sto19] ———, *Uniform bounds for the number of rational points on hyperelliptic curves of small Mordell-Weil rank*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 3, 923–956. MR3908770 ↑2

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