RECALL

$p > 2r+2$

$r < g$

$\boxed{\text{L-T, McP}}$ $\quad \#X(\mathbb{Q}) \leq \#\mathcal{X}^{sm}(\mathbb{F}_p) + 2g - 2$

$\boxed{\text{Stoll:}}$ Sps X hyperelliptic + $r \leq g-3$.

Then $$\#X(\mathbb{Q}) \leq 3(r+4)(g-1) + \max\{1, 4r\} \cdot g$$

$\boxed{\text{Katz-Rabinoff-ZB}}$ Sps $r \leq g-3$.

Then $$\#X(\mathbb{Q}) \leq 84g^2 - 98g + 28$$

RECALL

X $\boxed{\text{regular}}$ if $\forall\, p \in X$,

$$\begin{matrix} p \\ \updownarrow \\ \mathfrak{p} \end{matrix}$$

$$\dim_{k(\mathfrak{p})} \mathfrak{p}/\mathfrak{p}^2 = \dim_p X$$

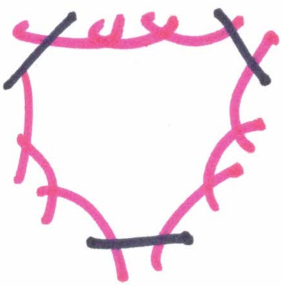
Regular $\Rightarrow$ $\operatorname{im}\left(\mathcal{X}(\mathbb{Q}_p) \to \mathcal{X}(\mathbb{F}_p)\right)$

$$\subseteq \mathcal{X}^{sm}(\mathbb{F}_p)$$

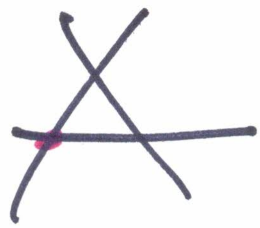
$$\underset{HL}{=}$$

Problem: $\mathfrak{X}^{sm}(\mathbb{F}_p)$ is unbounded?

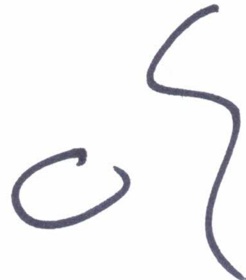
$$xyz = p^n(x^3+y^3+z^3)$$



3n-gon



locally $xy = p^n$



Stall: - work w/ non-regular model
- contract chains of $\mathbb{P}^1$'s

$\Rightarrow \mathfrak{X}(\mathbb{F}_p)$ is bounded
(via $p + g$)

Prob: Setup + local analysis is hard

Main tool (KRZB)

Systematic use of Berkovich & tropical geometry

Setup: 2 types of $\int$

$$\int_w^{AB} + \int_w^{BC}$$

- comes from Lie J
- $\int_p^{a} \omega^{AB} = 0$

- Computable
- not equal to $\int^{AB}$

Difference factors thru Trop $T \simeq \mathbb{R}^i$

$$\begin{array}{ccccc} & & T & & \\ & & \cong & & \\ \Lambda & \subseteq & \widetilde{J} & \longrightarrow & J^{an} \\ & & \downarrow & & \\ & & A & & \end{array}$$

A good red

Local analysis

potential thy on X^{an}

"Global" step:-

"combinatorial optimization" for sections of "Tropical canonical bundle"

Toy version:- $|K_\Gamma| \ni D$,

bound the min. # of firings which witness $D \sim K_\Gamma$

Time travel to AWS 2007

$$\mathcal{M}(R) = \{ R \xrightarrow{|\cdot|} \mathbb{R} \text{ bounded multiplicative seminorms} \}$$

$$\begin{array}{ccc} \downarrow \text{ev}_f & & \overset{|\cdot|}{\downarrow} \\ \mathbb{R} & & |f| \end{array}$$

$x \in \mathcal{M}(R)$, $f \in R$, "$f(x)$" $:= |f|$

Example: $\mathbb{Q}_p\{t\} \ni \sum_{i=0}^{\infty} a_i t^i$ $\quad a_i \in \mathbb{Q}_p$, $|a_i| \to 0$

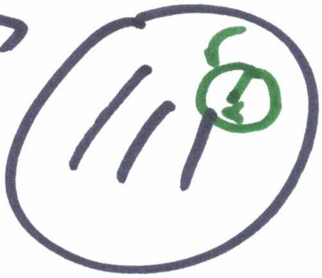
$$\mathcal{M}(\mathbb{Q}_p\{t\}) \ni |\cdot|_x$$

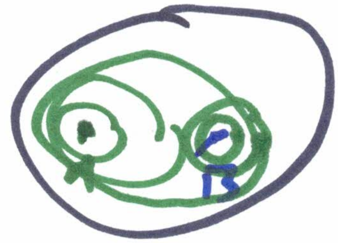
$$\cup\!\!\!| \qquad\qquad |f| = |f(x)|$$

Max Spec $\mathbb{Q}_p\{t\}$

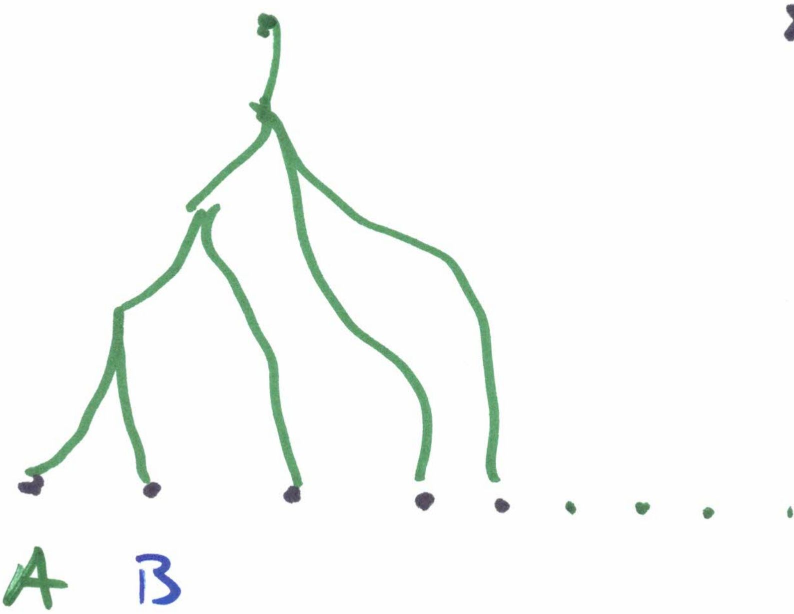
$$\overset{\psi}{x} \in \overline{\mathbb{Z}}_p \subset \overline{\mathbb{Q}}_p$$

$\mathbb{Z}_p$

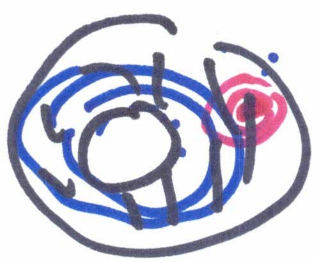


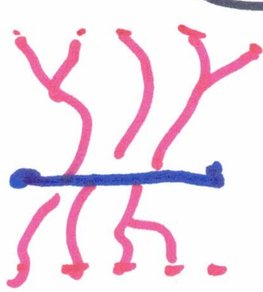


More pts: $|f|_{B(x,r)} = \sup_{x \in B} |f(x)|$



$xy = p^n$





Goal Glue, sheaves, coh, etc

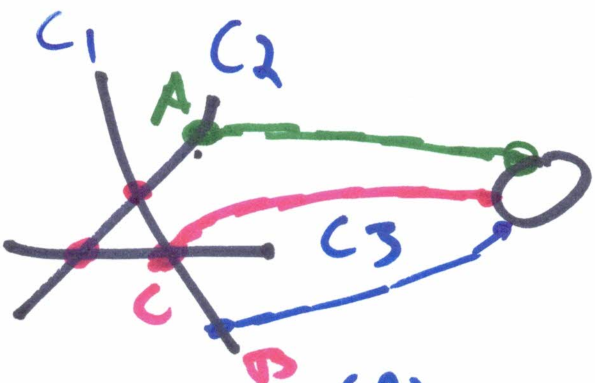
Fundamental THM of Trop. Geometry

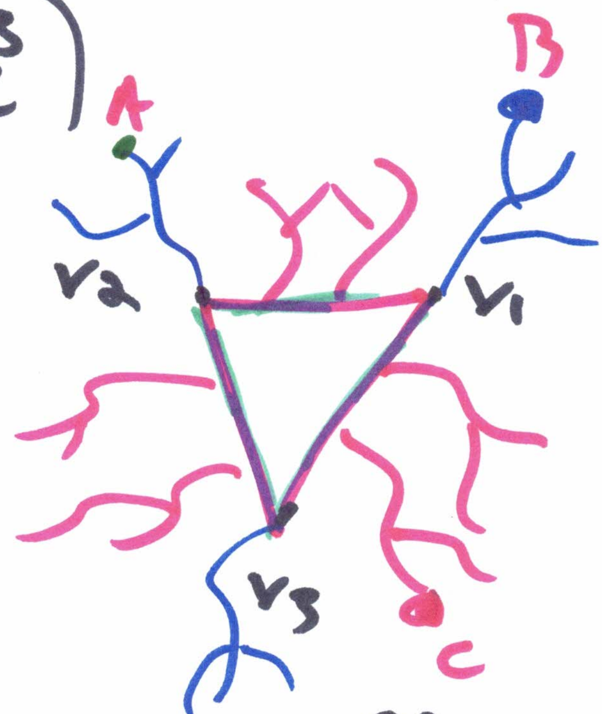
Baker–Payne–Rabinoff

Berkovich, Thuillier

$\mathfrak{X}/\mathbb{Z}_p$ s.stable

$xyz = p(x^3+y^3+z^3)$





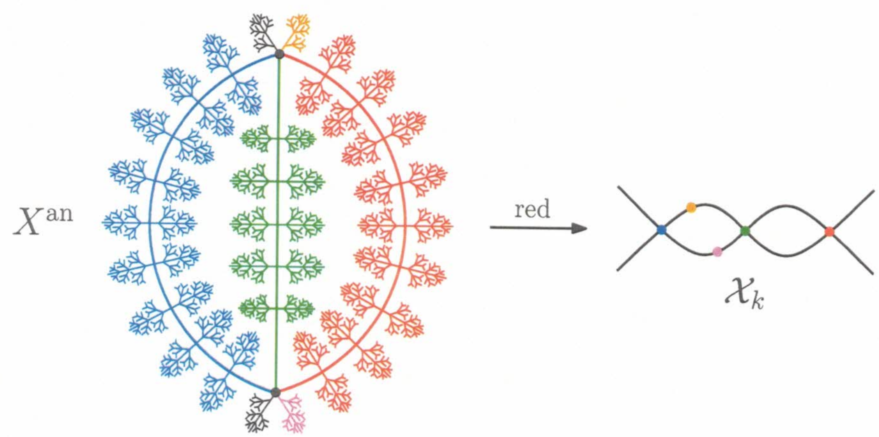
$|f| \mapsto v_{C_i}(f)$

$\mathfrak{X}_{\mathbb{F}_p}$ $\qquad$ $\mathfrak{X}_{\mathbb{Q}_p}$ $\qquad$ $\mathfrak{X}^{an}$

$\cup$ $\qquad$ $\exists\, r$ contraction

$\exists, \Gamma$ $\qquad$ +

compatible w/ reduction



X^{an}
red
$\mathcal{X}_k$

Potential THY $\mathcal{X}^{an} \xrightarrow{f} \mathbb{P}^1$

$F(x) := -\log|f(x)|$ $F|_\Gamma$ is p/w linear

THM: $r_* \operatorname{div} f = \operatorname{div} F$

$$:= \sum \begin{pmatrix} \Sigma \text{ in coming} \\ \text{slopes} \\ @ \\ P \end{pmatrix} [P]$$

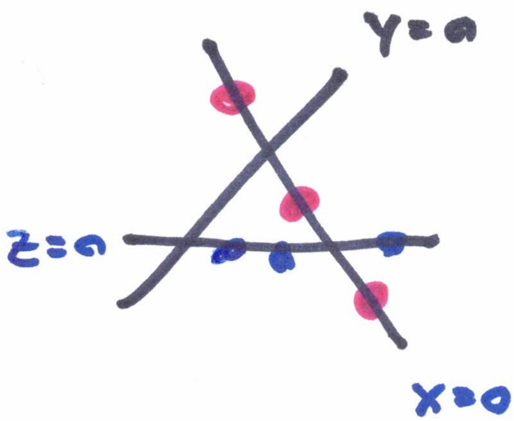
Exampl

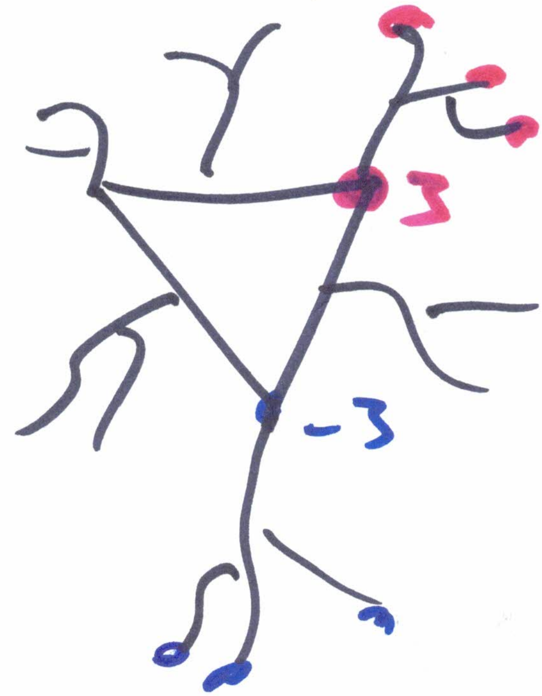
Example: $f(x:y:z) := [x:y:z]$

$$\operatorname{div} f = [0:\xi_0:1] + \dots - \left([1:\xi_0:0] + \dots \right)$$

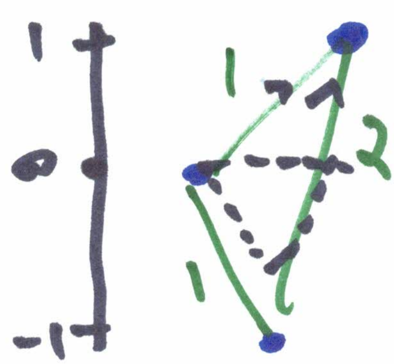
$y^3 + z^3 = 0$

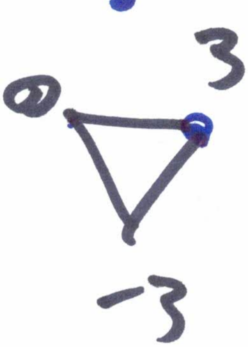
$x^3 + y^3 = 0$





$T_a \operatorname{div} f$





KRZB: $h, s, \|\cdot\|$

$$\operatorname{div} s = dd F + \|G\|$$

$$F = -\log |s|$$

(KR2B) Fa, $|\cdot|$ metrized L.B., S 7

$\tau_x \, d\mu_S = d\mu_F + \|G\|$
volume

$X = \mathcal{X}^{an}$

Axioms for integration

$\int : P(X) \times Z\Omega^1 \to \mathbb{C}_p$

- $X \overset{open}{\supseteq} U$ poly disc, $\omega|_U = df$

$$\int_\gamma df = f(\gamma(1)) - f(\gamma(0))$$

- $\int_\gamma \omega$ only depends on $[\gamma]$
- linear.

Still choices

BC

$X = \mathbb{G}_m^{an}$, $\omega = \frac{dT}{T}$ $\qquad \int \omega = \log T$

$\simeq *$

$H^1_{dR} \neq 0$

$$\mathbb{C}_p^* \supseteq \mathcal{O}_{\mathbb{C}_p}^* \supseteq \mathcal{O}_{\mathbb{C}_p}^{(1)} \xrightarrow{\log} \mathbb{C}_p$$

$p \in \mathbb{C}_p^*$, $T \in \mathcal{O}_{\mathbb{C}_p}^*$

gp homom

via Power series

Must choose $\log p$

<u>Date:</u> $\overline{\mathbb{C}_p^* / \mathcal{O}_{\mathbb{C}_p}^*} \simeq \mathbb{R} = \mathrm{Trop}\, \mathbb{G}_m$

(Calc III) Require Functorality $\begin{pmatrix} \text{Math} \\ \text{SS} \\ \text{Joke} \end{pmatrix}$ \9

$X \xrightarrow{f} Y$, $\omega \in Z^1_{dR}(Y)$

$\gamma \in P(X)$

$$\int_\gamma f^* \omega = \int_{f_* \gamma} \omega$$

Apply to $C \overset{i}{\hookrightarrow} J$ AJ

$$\int_\gamma i^* \omega = \int_{i_* \gamma} \omega$$

Apply to $[n]$ on J

$$\int_0^D \omega = \frac{1}{n} \int_0^{nD} \omega \qquad [n]^* \omega = n\omega$$

Call this $\int^{Ab}$, ~~and~~ use for CHAB

$$\int_P^Q \omega = 0 \text{ for } P, Q \in C(\mathbb{Q})$$

THM: (Coleman, de-Shalit, Berk, Dwork)

Fix log p, require Fun for Frob.

Then this uniquely determines an integral $\int^{BC}$ s.t.

$$\forall\ U \underset{\text{open}}{\subseteq} X^{an},\ \forall \gamma,\ \int_\gamma df = f(\gamma(1)) - f(\gamma(0))$$

↑ can be analic, or even "wide open" ⟷ Joe Pr.

IE, useful for computation.

But, $\int^{BC} \neq \int^{AB}$

IE $\int_P^Q \omega \neq 0$ for $P, Q \in X(\mathbb{Q})$

<u>THM (Coleman)</u>

If X has good reduction,
then $\int^{BC} = \int^{AB}$.

<u>Pf (KRZB)</u> X^{an} is contractible ($\Rightarrow$)

Jen B: $\displaystyle\int_{\mathrm{Frob}\,Q}^{\mathrm{Frob}\,R} \omega = \int_Q^R \mathrm{Frob}^* \omega$

"Analytic continuation along Frobenius"

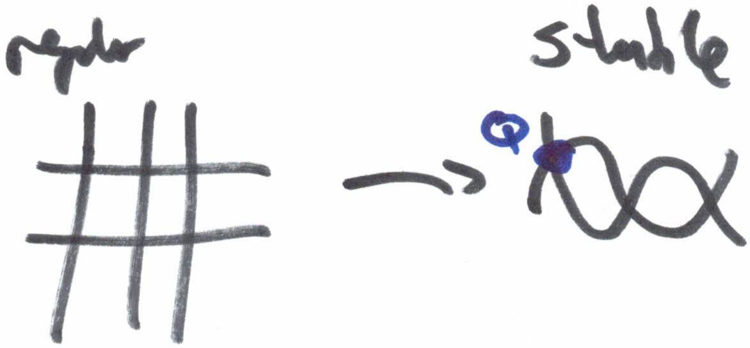
Final Remarks:

✓'d

why $r \le g-3$?

THM (Stoll): The difference

$$\int^{AB} \omega - \int^{BC} \omega \quad \text{is linear in } \omega$$

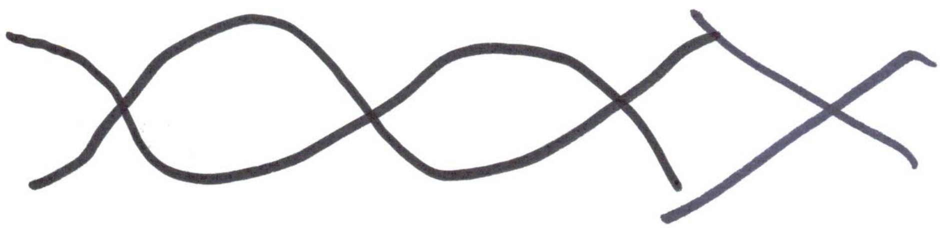


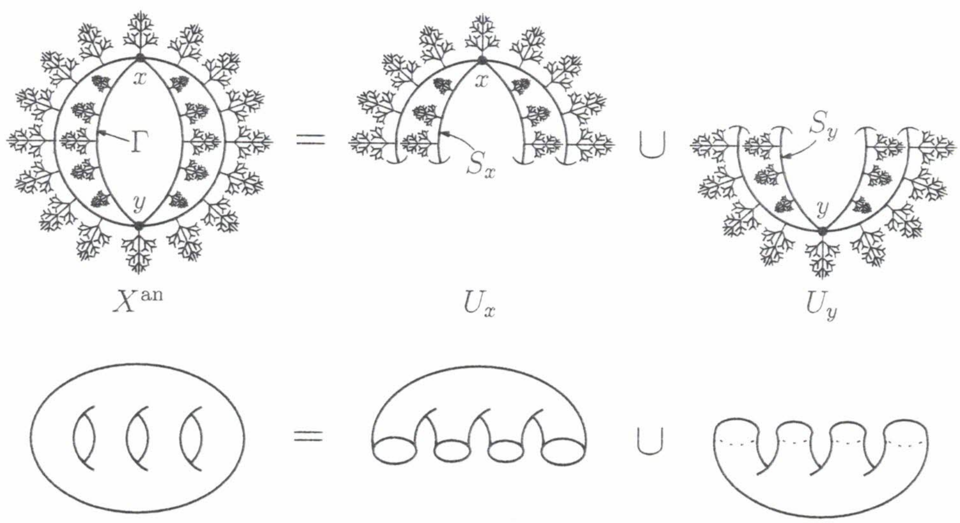
Chab in annuli

$$\omega|_{]Q[} = \sum a_i t^i = df + \frac{a_{-1}}{T}$$

not exact!

"Rolles"? Computing dN F $\leftrightarrow$ locally computing w/ N-polygons





x
y
Γ
=
S_x
S_y
∪
X^an
U_x
U_y