(Stoll) $p > 2g$, $r < g$, $X_{\mathbb{F}_p}$ smooth

$$\# X(\mathbb{Q}) \leq \# X(\mathbb{F}_p) + 2r$$

<u>Pf:</u>

$$\# X(\mathbb{Q}) \leq \sum_{Q \in X(\mathbb{F}_p)} (1 + n_Q)$$

$$= \sum 1 + \sum n_Q$$

$$= \# X(\mathbb{F}_p) + \deg D, \text{ where}$$

$$D = \sum n_Q [Q] \quad +$$

$$n_Q = \min_{\omega \in V} n_Q(\omega)$$

$$n_Q(\omega) := \mathrm{val}_Q(\tilde{\omega}) = \deg(\operatorname{div} \omega \cap \,]Q[\,)$$

$$V \subseteq H^0(X_{\mathbb{F}_p}, \Omega^1(-D))$$

$$\overset{\cup}{\omega} \iff \operatorname{div} \omega \geq D$$

$$g - r \leq \dim V \leq h^0(\Omega^1(-D))$$

$$\leq \frac{1}{2}(2g - 2 \mp \deg D) + 1$$

D & $K - D$ are special

Clifford: D special $\Longrightarrow$

$$h^0(D) \leq \frac{\deg D}{2} + 1$$

Ranks + linear series

$$|D| = \{E \geq 0 : E \sim D\} \simeq \mathbb{P}^n$$

$r(D) = -1$ if $|D| = \emptyset$

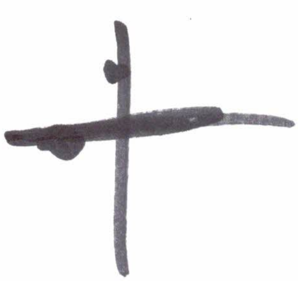
$r(D) \geq 0$ if $|D| \neq \emptyset$

$r(D) \geq 1$ if $\forall P \in X,\ |D-P| \neq \emptyset$

$r(D) \geq i$ if $\forall E \geq 0$ of deg $E \leq i$,
$|D-E| \neq \emptyset$

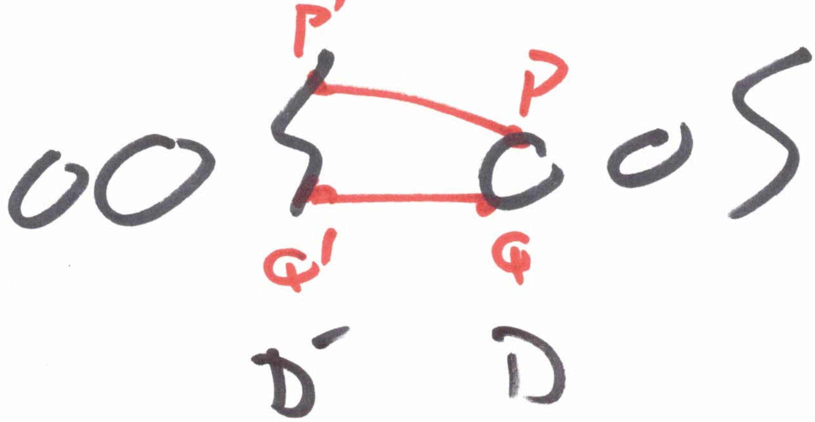
X sm $\Rightarrow r(D) = \dim H^0(X, D) - 1$

X singular $\not\Rightarrow r(D) \neq \dim H^0(D)$



rank of D is semicontinuous

$\mathcal{X}/\mathbb{Z}_p \qquad \mathcal{X}_{\mathbb{Q}_p} \rightsquigarrow \mathcal{X}_{\mathbb{F}_p}$



$r(D) \le r(D')$ H III.12

<u>Can be strict</u>

$P, Q \in X$ not hyper

reducing to $P', Q' = i(P')$

$h^0(P+Q) = 1$

$h^0(P' + i(P')) = 2$

<u>Defn:</u> Let X be a ~~Sbe~~ scheme.

X is <u>regular</u> if

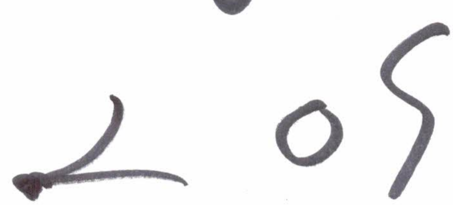
$\forall\, p \in X$, $\dim_{k(p)} \mathfrak{p}/\mathfrak{p}^2 = \dim X$

$$p \mapsto \mathfrak{p}$$

<u>Example</u>: $y^2 - x^3 - p \,/\, \mathbb{Z}_p$

$$\mathfrak{X} \xrightarrow{\text{not sm}} \operatorname{Spec} \mathbb{Z}_p$$

$\mathfrak{X}$ is regular.



$(0,0)$

$M = (x, y, p)$

$M/M^2 = \langle x, y, p \rangle$

$= \langle x, y \rangle_{\mathbb{F}_p}$

$p \in M^2$

$\mathfrak{X}/\mathbb{Z}_p$ regular proper model of a sm. curve

$$\mathrm{im}\left(\mathfrak{X}(\mathbb{Q}_p) \xrightarrow{\text{red}} \mathfrak{X}(\mathbb{F}_p)\right)^{(*)} \subseteq \mathfrak{X}^{sm}(\mathbb{F}_p)$$

<u>Example</u> $y^2 - x^3 - p \qquad]0,0[\, = \emptyset$

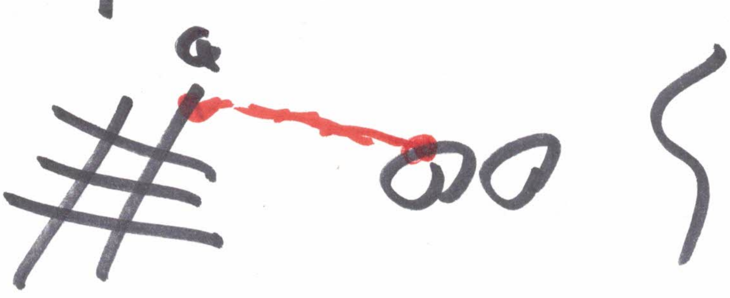
$y^2 - x^3 - p^2 \qquad]0,0[\, \ni (0,p)$

Lorenzini-Tucker, McPoonen

$r < g$ etc, $\mathfrak{X}$ r. p. model

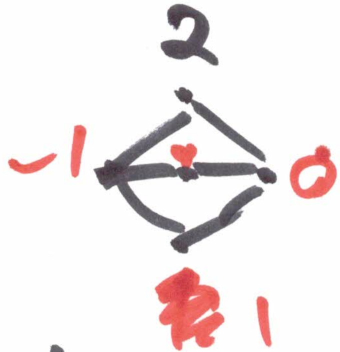
$$\#X(\mathbb{Q}) \le \#\mathfrak{X}^{sm}(\mathbb{F}_p) + 2g - 2$$

"Same proof" $\qquad]Q[\, \cong p\mathbb{Z}_p$



M. Baker's Idea: degenerate even more

$\xrightarrow[\text{graph}]{\text{dual}}$



cpt $\longrightarrow$ vtx $\qquad \mathbb{Z}^{V(\Gamma)}$

int $\longrightarrow$ edge $\qquad \shortparallel$

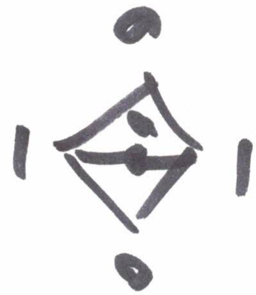
$D \longmapsto sp(D) \in \mathrm{Div}\,\Gamma$

Baker, Baker–Norine

Notion of $|D|$, $r(D)$

$r(D) \leq r(sp(D))$

$K_\Gamma = \sum (\deg v - 2)[v]$



Notion of special

RR + Cliff —

RR $r(D) - r(K_p - D) = \deg D + 1 - g$

Cliff D special $\Longrightarrow$

$$r(D) \leq \frac{\deg D}{2}$$

<u>THM</u> (KZB) $r < g$, $\mathfrak{X}$ rpm

Sps $\mathfrak{X}_{\mathbb{F}_p}$ is totally degenerate

then $sp(D_{chab})$ is special

$$D_{cm} = \sum n_Q [Q].$$

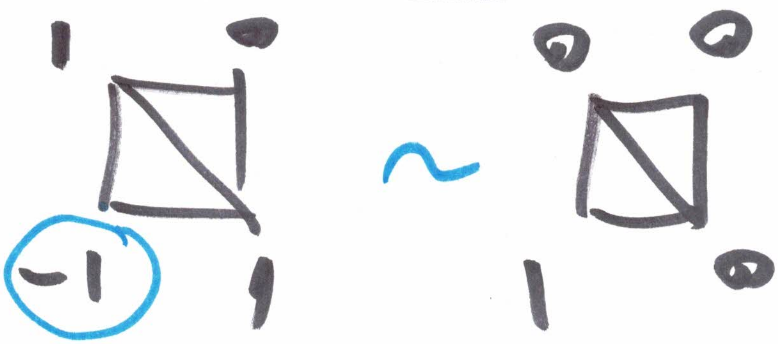
Actually, $r(K_p - D) \geq g - r - 1$

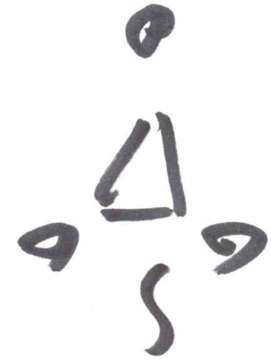
<u>Pf:</u> (of rank favorability)

$$g-r-1 \le r\left(K_\Gamma - sp(D)\right) \le \frac{\deg(K_\Gamma - \deg D)}{2}$$

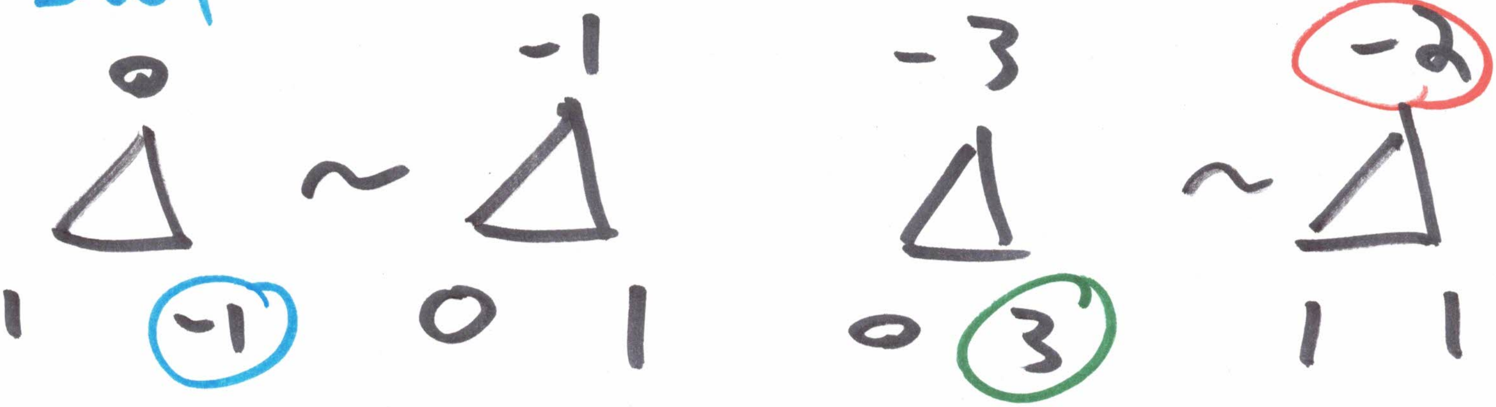
$$= \frac{1}{2}\left(2g-2+\deg D\right) \qquad \blacksquare$$

Chip firing





get out of Debt



$D \sim D'$ if they differ by some seq. of lends + borrows.

$\operatorname{Pic}\Gamma = \operatorname{Div}\Gamma/\sim$

$\cup|$

$\operatorname{Pic}^0\Gamma \simeq$ crit gp sandpile réseau mob

$|D| = \{ D' \geq 0 : D' \sim D \}$

$r(D) \geq i$ iff $\forall E \geq 0$ w/ $\deg E \leq i$, $|D - E| \neq \emptyset$

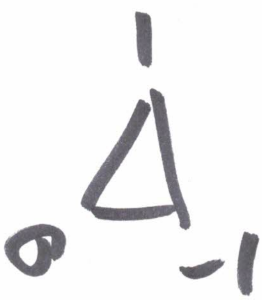
$r\Bigg($ 

$\Bigg) = 0$

$r\Bigg($ 

$\Bigg) = 1$

$\sim$ 

$\Bigg|$ 

$\Bigg| = \emptyset$

$\mathfrak{X}/\mathbb{Z}_p \quad \mathfrak{X}_{\mathbb{F}_p} = \bigcup C_i \quad \#_{C_1 C_2} \infty^{10}$

$$\mathrm{Pic}\,\mathfrak{X} \xrightarrow{sp} \mathrm{Div}\,\Gamma$$

$$h \longmapsto sp(h) :=$$

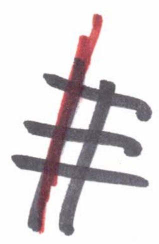
$$\sum \left(\deg h|_{C_i}\right)[x_i]$$

Examples: $h = \mathcal{O}(C_i)$

$$\deg \mathcal{O}(C_i)\big|_{C_j} = \begin{cases} \#\text{ int. pts of } C_i \cap C_j & i \neq j \\ \#\text{ self int. of } C_i & i = j \end{cases}$$

$= -\,\#$ int. pts of C_i w/ rest of $\mathcal{X}_s$

$$\mathcal{X}_{\mathbb{F}_p} = (p) \Rightarrow C_i \sim -\sum_{j \neq i} C_j$$



C_1

$sp(\mathcal{O}(C_i))$

$sp(\mathcal{O}(C_i)) \longleftrightarrow$ firing @ vertex i

Example $\mathcal{F} = \omega_{\mathfrak{X}}$

Adjunction:

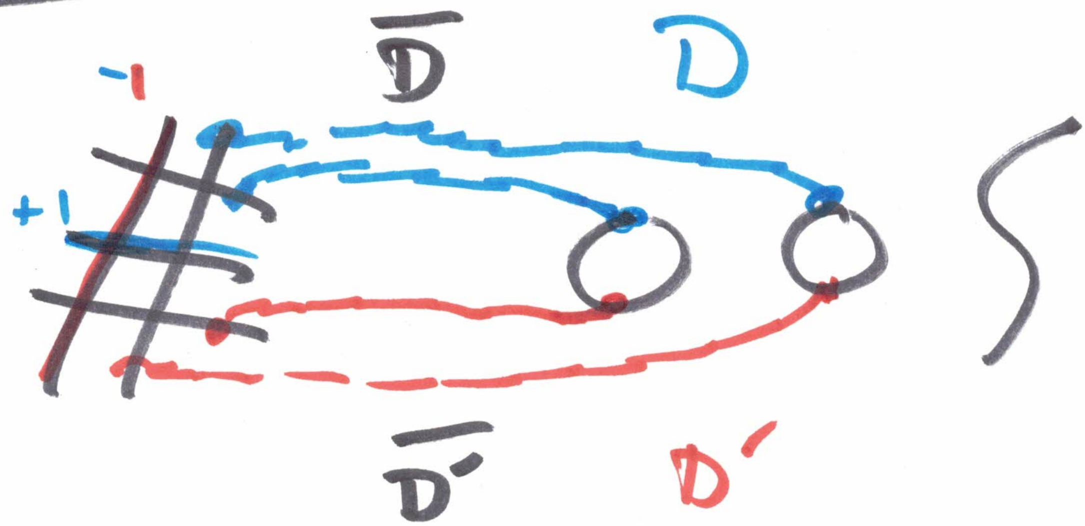
$$\left(\omega_{\mathfrak{X}} \otimes \mathcal{O}(C_i)\right)\Big|_{C_i} \simeq \Omega^1_{C_i}$$

$$\deg \omega_{\mathfrak{X}}|_{C_i} = \deg \Omega^1_{\mathbb{P}^1} - \deg \mathcal{O}(C_i)|_{C_i}$$

$= -2 + \#$ of in. pts of C_i w/ rest of $\mathfrak{X}$

$= k_i$

upshot



$D \sim D'$ on $\mathcal{X}_{Gp}$

$\operatorname{div} f = D - D'$ $\quad f : \mathcal{X}_{Gp} \to \mathbb{P}^1$

Extend f to $\mathcal{X} \hookleftarrow F$

Then $\operatorname{div} F = \overline{D} - \overline{D}' + \sum a_i C_i$

The equivalence $sp(\overline{D})$ w/ $sp(\overline{D}')$ is witnessed by bending @ C_i (a_i

a_i many times

$\omega \in \Omega^1_{\mathrm{Zar}}$

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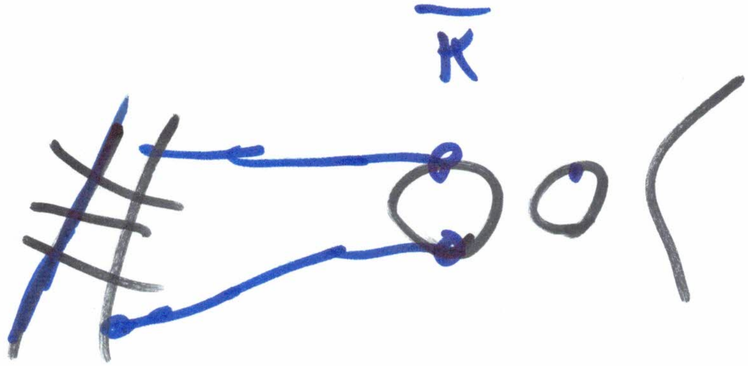

$\operatorname{div} \omega = \overline{K} + \sum b_i C_i$

sp $K \in |K_\eta|$ + ~~B.I.~~

firms @ b_i times @ v_i witness

sp $K \sim K_\eta$

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