$$
\begin{tikzcd}
 & & H^0(X_{\mathbb{Q}_p}, \Omega^1)^\vee \arrow[d, phantom, "\cong" rotate=-90] \\
 & & H^0(J_{\mathbb{Q}_p}, \Omega^1)^\vee \arrow[d, phantom, "\wr\!\mathrm{S}"] \\
X(\mathbb{Q}) \arrow[r] \arrow[d] & X(\mathbb{Q}_p) \arrow[d] \arrow[dr] & \\
J(\mathbb{Q}) \arrow[r] & J(\mathbb{Q}_p) \arrow[r, "\log"] & \operatorname{Lie} J_{\mathbb{Q}_p}
\end{tikzcd}
$$

$$D \longmapsto \left(\omega \longmapsto \int_0^D \omega\right)$$

Setup: $\exists\, V \subseteq H^0(X_{\mathbb{Q}_p}, \Omega^1)$ s.t.

$\dim V \geq g-r$ +

$\forall\, P, Q \in X(\mathbb{Q})$, $\qquad \displaystyle\int_P^Q \omega = 0$

$\forall\, \omega \in V$

$\int$'s are locally analytic

$P \in X(\overline{\mathbb{F}}_p)$

$Q_1, Q_2 \in]P[\xrightarrow{\sim} p\mathbb{Z}_p$

$Q \longmapsto t(Q)$

t unit @ P

$$\int_{Q_1}^{Q_2} \omega = \int_{t_1}^{t_2} f(t)\,dt = \sum \frac{a_i t^{i+1}}{i+1}\Bigg|_{t_1}^{t_2} = I_\omega(t)$$

$\boxed{\text{FTC Calc I}}$

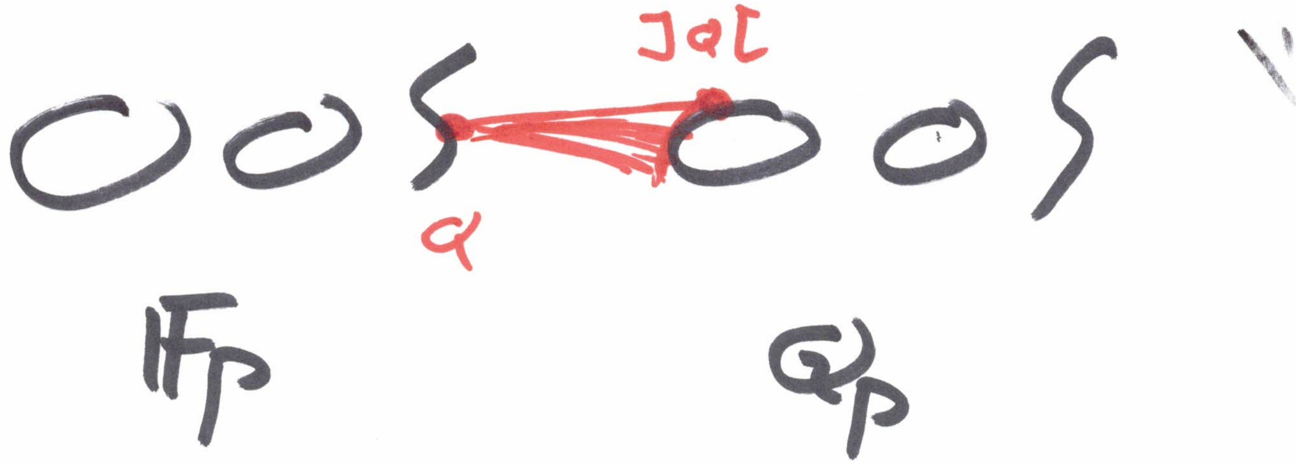
$\omega|_{]P[} = f(t)\,dt = \sum a_i t^i \qquad a_i \in \mathbb{Z}_p$

Coleman's THM $X/\mathbb{Q}$ nice, $r < g$

p good prime $p > 2g$

Then $\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2$

Pf.



Let $Q \in X(\mathbb{F}_p)$. Let $\omega \in V$.

Let $n_Q = \deg(\operatorname{div} \omega \cap]Q[)$.

Then $\#\{z \in p\mathbb{Z}_p \text{ s.t } I_\omega(z) = 0\}$ (*)

$\leq 1 + n_Q$.

THEN

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{Q}_p),$$

$$\leq \sum_{Q \in X(\mathbb{F}_p)} (1 + n_Q)$$

$$= \sum_{Q \in X(\mathbb{F}_p)} 1 + \sum_{Q \in X(\mathbb{F}_p)} n_Q$$

$$\leq \#X(\mathbb{F}_p) + 2g - 2 \quad \blacksquare$$

deg ω RR

Lemma (Coleman)

$\sum a_i t^i$ $\quad$ 3

Let $f(t) \in \mathbb{Q}_p[[t]]$ s.t. $f'(t) \in \mathbb{Z}_p[[t]]$.

$$= \sum \frac{a_i}{i+1} t^{i+1}$$

Let $m = \mathrm{ord}_{t=0}\left(f'(t) \bmod p\right)$.

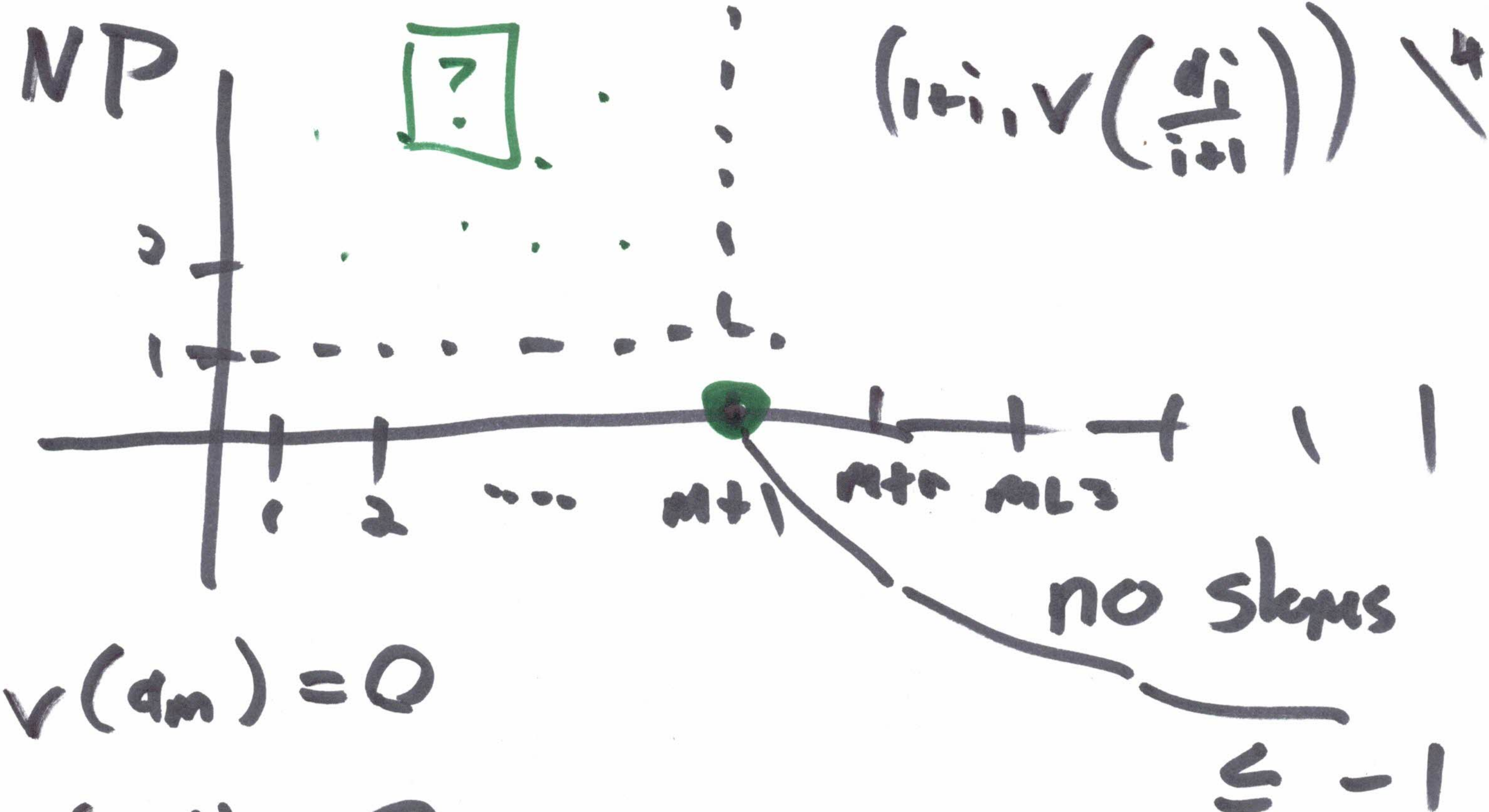
(Note: $m = n_Q$)

Sps $m < p-2$.

$$\boxed{\begin{array}{l} p > m+2 \\ m \le 2g-2 \\ RR \end{array}}$$

Then f has @ most $m+1$ zeroes in $p\mathbb{Z}_p$.

Proof. Newton Polygons. ∎



$v(a_m) = 0$

$v(m+1) = 0$

$v(a_i) \geq 0$ for $i < m$

$v(i+1) = 0$

$v\left(\frac{a_i}{i+1}\right) \geq -\cancel{v_p}(i+1)$

$-v_p(i+1) > m+1-(i+1)$

∎

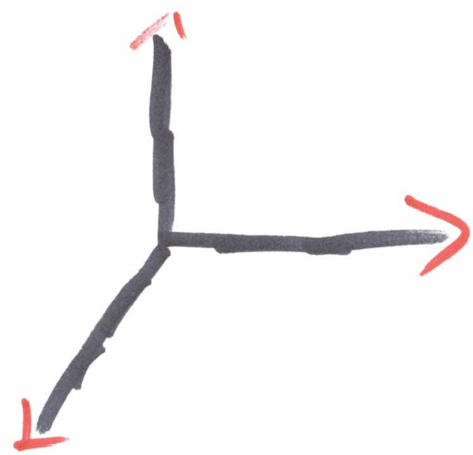
~~steps~~ Segments of slope α

~~Seg~~ $\leftrightarrow$ roots w $v(-) - \alpha$

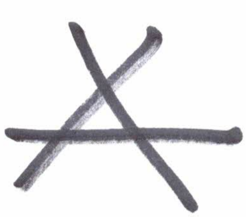
$X / \mathbb{Q}_p$ variety

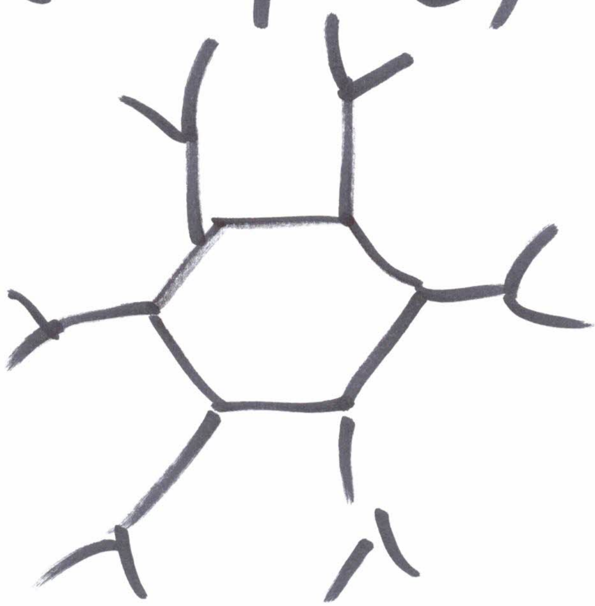
$$\mathrm{Trop}\, X := \overline{v(X(\bar{\mathbb{Q}}_p))}$$

$x + y = 1$



$\underline{xyz} = p(x^3 + y^3 + z^3)$





Stoll's idea:

Pick the "best" ω for each $Q \in X(\mathbb{F}_p)$.

Let $n_Q(\omega) = \deg\left(\operatorname{div} \omega |_{]Q[}\right)$

$$n_Q = \min_{\omega \in V} n_Q(\omega)$$

$$\# X(\mathbb{Q}) \leq \sum_{Q \in X(\mathbb{F}_p)} (1 + n_Q)$$

$$\leq \# X(\mathbb{F}_p) + \sum_{Q \in X(\mathbb{F}_p)} n_Q$$

$$\sum n_Q \le \sum n_Q(\omega) \le 2g-2$$

$$D := \sum_{Q \in X(\mathbb{F}_p)} n_Q [Q]$$

<u>Claim:</u> $\deg D \le 2r$

$\dim V \ge g - r$

<u>Obs:</u> D is special

$K_{\exists D}$ canonical div

D is <u>special</u> if

$\dim H^0(X, K-D) \geq 1$

$\Longleftrightarrow$

$\exists$ some canonical divisor $K' \geq 0$

s.t. $D \leq K'$

$\dim H^0(X, \Omega^1(-D)) > 0$

$0 \neq \overset{\omega}{w} \longleftrightarrow \operatorname{div} w \geq D$

THM (Clifford)

If D is special,

then ~~$\deg D \leq$~~

$$\dim H^0(X, D) \leq \frac{\deg D}{2} + 1$$

Context: RR

$$h^0(D) - h^0(K-D) = \deg D + 1 - g$$

Pf. $D = \sum n_Q [Q]$

$V \subseteq H^0(X_{\mathbb{F}_p}, \Omega^1(-D))$

$\Rightarrow$

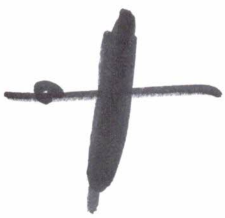
$g - r \leq \dim H^0(\Omega^1(-D))$

$\leq \frac{1}{2}(\boxed{2g} - 2 - \deg D) + 1$

$\Rightarrow \deg D \leq 2r$ $\square$

The notion of special is

not good to singular curves



$H^0(D) \ni f$

K is often not special

$\exists$ curves

s.t. $\nexists$ an effective can. divisor