Coleman "Effective Chabauty"

Lorenzini - Tucker

McCallum - Poonen

Stoll

Katz - ZB

Let $X/\mathbb{Q}$ be a <u>nice</u> curve

$$r = \text{rank } J(\mathbb{Q})$$

THM (KZB) Let $p > 2r+2$ prime

Let $\mathcal{X}$ be a regular proper minimal model of X.

Let $r < g$.

Then

$$\# X(\mathbb{Q}) \leq \# \mathcal{X}^{sm}_{\mathbb{F}_p}(\mathbb{F}_p) + 2r.$$

Coleman: $\cancel{2r}$ $2g-2$

"rank favorable"

Mazur: Can we bound

$\# X(K)$ via rank $J(K)$?
$+ g$

Uniformity conj

$\exists B(K, g)$ s.t. $\forall$ nice X/K of genus g,

$$\# X(K) \leq B(K, g)$$

Poonen et al

Heuristics $\Rightarrow r$ is bounded

Weak Lang Conj:

Let X/K be a variety of General type.

Then $\exists\, Z \subsetneq X$ s.t.
closed

$$Z(K) \supseteq X(K)$$

<u>THM</u> (Caparso, Harris, Mazur)

$$WLC \Rightarrow UC$$

Example (Gordon - Grant) '93

$X : y^2 = x(x-1)(x-2)(x-5)(x-6)$

$g = 2$

$r = 1 < 2$

$\#X(\mathbb{Q}) = 10$ 　 6 WP

$(3, \pm 6)$ &

$(10, \pm 120)$

$\#X(\mathbb{F}_7) = 8$ 　 6 WP

$(3, \pm 6)$

$$10 \leq \#X(\mathbb{Q}) \leq \#X(\mathbb{F}_7) + 2$$

$$= 8 + 2 = 10$$

THM (Stoll) Sps X is hyperelliptic.

Sps $r \leq g-3$.

then $\#X(\mathbb{Q}) \leq 3(r+4)(g-1) + \max\{1, 4r\} \cdot g$

THM (Katz-Rabinoff-ZB)

Sps $r \leq g-3$. then

$\#X(\mathbb{Q}) \leq 84g^2 - 98g + 28$.

$(*)$

Effective Manin - Mumford

X curve

$X \xhookrightarrow{i} J$

$\# \, i(X) \cap J_{tor} < \infty$ Raynaud, Buium, Coleman

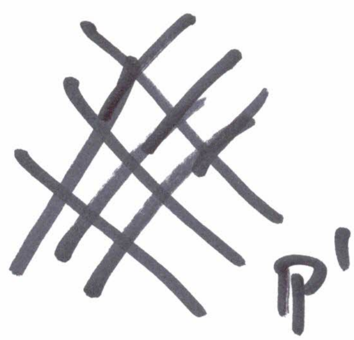
$X \hookrightarrow J \xrightarrow{\log} \mathrm{Lie}\, J$

$\Rightarrow$ $X \cap J_{tors}$ ← integrals vanish on

no nessessory rank cond.

(KRZB)

- $(X \cap J_{trop})(\mathbb{Q}) \leq (*)$
- If X is very degenerate (eg totally degenerate)



AND trivalent dual graph
eg

then we can bound

$\# X \cap J_{trop} \leq$ ——

$$
\begin{tikzcd}
X(\mathbb{Q}) \arrow[r, hook] \arrow[d] & X(\mathbb{Q}_p) \arrow[d] \arrow[dr] & \\
J(\mathbb{Q}) \arrow[r, hook] & J(\mathbb{Q}_p) \arrow[r, "\log"] & \operatorname{Lie} J
\end{tikzcd}
$$

Black box Chabauty

- Setup
- local analysis
- global coordination

Setup $r < g$

$\exists V \subseteq H^0(X_{\mathbb{Q}_p}, \Omega^1)$ s.t.

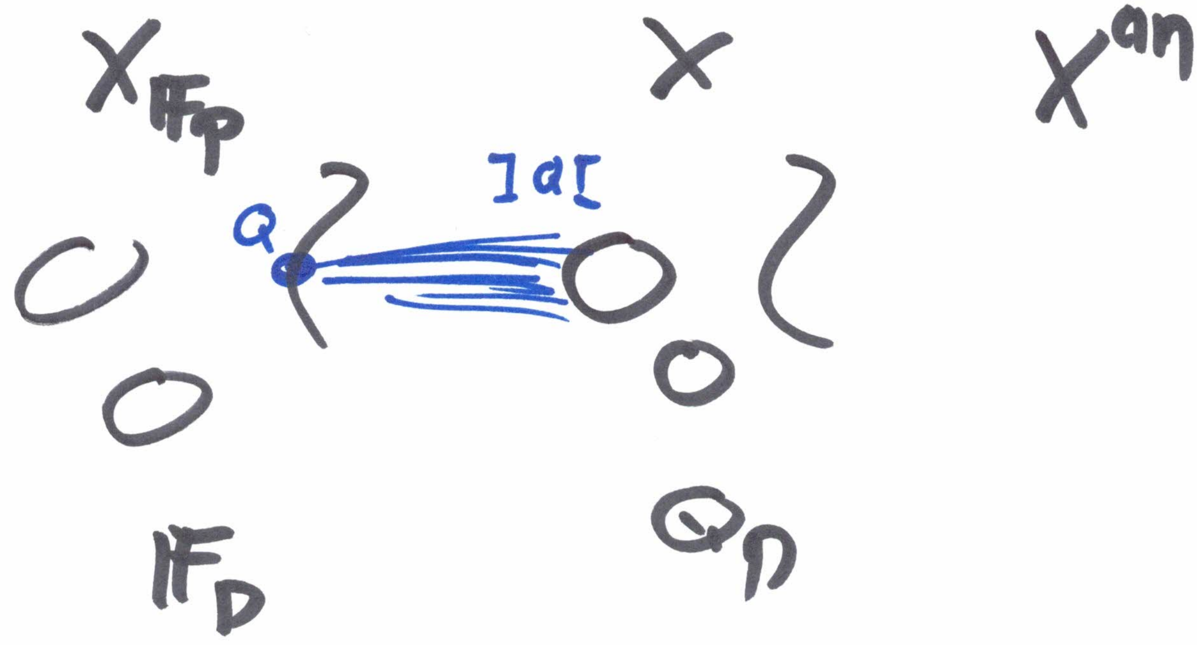
$\forall P, Q \in X(\mathbb{Q}),$
$\forall \omega \in V$ $\quad \int_P^Q \omega = 0$

$+ \dim V \geq g - r > 0$

Local analysis:

we can compute $\int_P^Q \omega$ locally +

Analyze w/ eg Newton Polygons



$]Q[:= \{ P \in X(\mathbb{Q}_p) \text{ s.t. } P \equiv Q \bmod p \}$

$\cong p\mathbb{Z}_p$ p-adic disc.

$P \longmapsto u(P)$

u uniformizer @ $\tilde{Q}$

$\uparrow$ lift of Q

Example: (MP survey)

$$X:\ y^2 = f(x) = x^6 + 8x^5 + \cdots + 1$$
$$= x\,g(x) + 1$$

$$(0,1) \in X(\mathbb{F}_3)$$

$$]\,(0,1)\,[\ \cong p\mathbb{Z}_p$$

$$\overset{4}{P} \longmapsto X(P)$$

$$\cancel{(x,y)}$$

$$\left(t, \sqrt{t g(t) + 1}\right) \longleftarrow\!\shortmid\ t$$

↑

converges b/c t is small

$$v_p(t) > 0$$

To compute $\int_P^Q \omega$, only compute

"tiny" integrals, i.e.

$P \equiv Q \bmod p$. $\quad P, Q \in X(\mathbb{Q})$

If $Q \in]\tilde{P}[\cong p\mathbb{Z}_p \ni t$

$$\omega\big|_{]P[} = f_\omega(t)\,dt$$

for some $f(t) \in \mathbb{Z}_p[[t]]$

$$\int_P^Q \omega = \int_0^{t_Q} f_\omega(t)\,dt$$ ← integrate formally.

$$= I(t)$$

p-adic Rolle's thm: $\boxed{p > 2r+3}$

relates # of 0's of $I(t) = 1 + n_p$

to # of 0's of $f_\omega(t) = n_p$

Pf: Next time

(essentially NP)