Equations for Selmer varieties: project brief

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Let S be a finite set of (rational) primes and $\mathcal{X}/\mathbb{Z}_S$ a suitable model of a hyperbolic curve $X/\mathbb{Q}$. A central role in the non-abelian Chabauty method is played by the (global) *Selmer variety* $\mathrm{Sel}_{S,U}(\mathcal{X})$ [1, §2 & §8] corresponding to a suitable quotient U of the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of X (at a rational basepoint). Whenever the localisation map $\mathrm{loc}\colon \mathrm{Sel}_{S,U}(\mathcal{X}) \to \mathrm{H}^1_f(G_p, U)$ is non-dominant – for instance for dimension reasons – the set $\mathcal{X}(\mathbb{Z}_S)$ of S-integral points is finite. More precisely, each defining equation of the (scheme-theoretic) image of the localisation map gives rise to a p-adic analytic function on $\mathcal{X}(\mathbb{Z}_p)$ vanishing on $\mathcal{X}(\mathbb{Z}_S)$, and by understanding these functions one can often compute the set $\mathcal{X}(\mathbb{Z}_S)$ in practice.

The broad aim of this project is to understand the equations cutting out the image of the localisation map. There are several key examples in the literature where the non-abelian Chabauty method has been made explicit (to varying extents), including:

1. the quadratic Chabauty method [2, 3];
2. explicit Chabauty–Kim for $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ [4, 5]; and
3. the contributions from places $\ell \in S$ [6].

The first part of this project will consist of collating these examples and recasting them in the language of equations for Selmer images.

Once these examples have been collated, the second part of the project will be to explore several new examples where explicit equations for Selmer varieties can be found, and then to use these to determine rational or S-integral points. The main example we will consider is determining S-integral points on $\mathcal{X} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ for $\#S = 2$. Using the refined Selmer varieties of [6], one expects the Selmer image to be cut out by quadratic relations in depth 2 (i.e. for the 2-step unipotent quotient U_2), and we will identify exactly what these relations are. Using this, we will compute $\mathcal{X}(\mathbb{Z}_p)_{S,U_2}$ in small cases (esp. $S = \{2, 3\}$), and compare it to $\mathcal{X}(\mathbb{Z}_S)$.

Project requirements

You should already have a basic understanding of the workings of non-abelian Chabauty and the theory of Selmer varieties: what they are and how one controls their dimension. An understanding of iterated Coleman integration is helpful for the latter parts of the project, but not necessary. You should be willing to familiarise yourself with the contents of at least one of the papers [2, 3, 4, 5, 6] before the winter school and present the main results to the rest of the group.

References

[1] J. Balakrishnan, I. Dan-Cohen, M. Kim, S. Wewers: *A non-abelian conjecture of Tate-Shafarevich type for hyperbolic curves*, Mathematische Annalen, 372(2018), no. 1-2, 369–428. arXiv:1209.0640 (v4)

[2] J. Balakrishnan, N. Dogra: *Quadratic Chabauty and rational points I: p-adic heights* (with appendix by J.S. Müller), Duke Mathematical Journal, 167(2018), no. 11, 1981–2038. arXiv:1601.00388 (v2)

[3] J. Balakrishnan, N. Dogra: *Quadratic Chabauty and rational points II: Generalised height functions on Selmer varieties*, International Mathematics Research Notices (to appear). arXiv:1705.00401 (v2)

[4] I. Dan-Cohen, S. Wewers: *Mixed Tate motives and the unit equation*, International Mathematics Research Notices, (2016), no. 17, 5291–5354. arXiv:1311.7008 (v3)

[5] I. Dan-Cohen: *Mixed Tate motives and the unit equation II*, Algebra and Number Theory (to appear). arXiv:1510.01362 (v2)

[6] L.A. Betts, N. Dogra: *The local theory of unipotent Kummer maps and refined Selmer varieties.* arXiv:1909.05734 (v2)