Selmer Schemes II, III

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Disclaimer

These lecture slides come with a bibliography at the end. However, there has been no attempt at accurate attribution of mathematical results. Rather, the list mostly contains works the lecturer has consulted during preparation, which he hopes will be helpful for users.

I. De Rham fundamental groups

De Rham fundamental groups

F: a finite extension of $\mathbb{Q}_p$.

X: a smooth curve over F

$\bar{X}$: the basechange of X to $\bar{F}$.

$b, x \in X(F)$ viewed sometimes as geometric points:

$$\mathrm{Spec}(\bar{K}) \longrightarrow \bar{X} \longrightarrow X.$$

$\mathcal{X}$: a smooth scheme over $\mathcal{O}_F$, the valuation ring of F, with good compactification and generic fiber X.

Y:special fiber of $\mathcal{X}$ over $k = \mathcal{O}_F/m_F$.

De Rham fundamental groups

The De Rham version is similar to the etale case [Hain, AIK, Kim3]. The relevant category is

$$\mathrm{Un}^{DR}(X) \subset \mathrm{Loc}^{DR}(X)$$

the category of unipotent vector bundles with (flat) connections, a full subcategory of all bundles with flat connections.

There are fibre functors

$$F_b : \mathrm{Un}^{DR}(X) \longrightarrow \mathrm{Vect}_F,$$

$$(V, \nabla) \mapsto V_x$$

and the objects of interest are

$$U^{DR} = U^{DR}(X, b) = \mathrm{Aut}^{\otimes}(F_b)$$

and

$$P^{DR}(x) = P^{DR}(X; b, x) = \mathrm{Isom}^{\otimes}(F_b, F_x)$$

De Rham fundamental groups

They can be constructed using universal objects which in turn admit a tautological construction [AIK] using

$$\mathrm{Ext}^i_{\mathrm{Loc}^{DR}(X)}((V,\nabla),(V',\nabla')) \simeq H^i_{DR}(X,(V,\nabla)^* \otimes (W,\nabla)),$$

where

$$H^i_{DR}(X,(V,\nabla)) = H^i(X_{Zar}, V \longrightarrow V \otimes_{\mathcal{O}_X} \Omega_X)$$

In particular, it is a projective system

$$(\mathcal{E}^{DR}_n, \nabla_n),$$

which fit together as

$$0 \longrightarrow T^{DR}_n \otimes \mathcal{O}_X \longrightarrow \mathcal{E}^{DR}_n \longrightarrow \mathcal{E}^{DR}_{n-1} \longrightarrow 0.$$

Here, T^{DR}_n is a quotient of $(H^{DR}_1)^{\otimes n}$ as in the étale case.

De Rham fundamental groups

After choosing an element $1 \in \mathcal{E}_b^{DR}$ we get the universal property:

Given any object (V, ∇_V) in $Un^{DR}(X)$ together with an element $v \in V_b$ (the fiber at b), there exists a unique morphism $\phi : (\mathcal{E}^{DR}, \nabla) \to (V, \nabla_V)$ such that $1 \in \mathcal{E}_b^{DR} \mapsto v$.

Corollary

$$End(F_b) \cong \mathcal{E}_b^{DR}.$$

De Rham fundamental groups

Theorem

The pro-algebraic group $U^{DR}(X, b)$ is isomorphic to the group-like elements in $\mathcal{E}_b$, while $P^{DR}(X; b, x)$ is isomorphic to the group-like elements in $\mathcal{E}_x$.

The universal property gives rise to a map in $\mathrm{Un}(X)$:

$$\Delta : (\mathcal{E}^{DR}, \nabla) \longrightarrow (\mathcal{E}^{DR}, \nabla)\hat{\otimes}(\mathcal{E}^{DR}, \nabla)$$

that takes 1 to $1 \otimes 1$.

Let $\mathcal{A}^{DR} = \mathcal{E}^{DR}$ be the dual (ind-)bundle. Then Δ^* gives

$$\mathcal{A}_x^{DR} = \mathrm{Hom}(\mathcal{E}_x^{DR}, K)$$

the structure of a commutative algebra, and

$$P^{DR}(x) = \mathrm{Spec}(\mathcal{A}_x^{DR}).$$

De Rham fundamental groups: Hodge filtration

[Hain, Wojtkowiak, Vologodsky, Hadian, Kim3]

There is a unique decreasing filtration $\mathcal{F}^i$, $i \leq 0$, of $\mathcal{E}^{DR}$ satisfying the following conditions.

(1) Griffiths transversality $\nabla(\mathcal{F}^i) \subset \mathcal{F}^{i-1} \otimes \Omega_X$;

(2) The induced filtration on T_n coincides with the constant one coming from (co)homology;

(3) $1 \in F^0\mathcal{E}_b^{DR}$.

This is the Hodge filtration on $\mathcal{E}^{DR}$.

There is an induced Hodge filtration with non-negative degrees on $\mathcal{A}^{DR}$ and $F^1\mathcal{A}^{DR}$ is an an ideal. $F^0P^{DR}(x)$ is the defined to be the zero set of $F^1\mathcal{A}_x^{DR}$. It is a torsor for F^0U^{DR}, which is a subgroup of U^{DR}.

De Rham fundamental groups: Hodge filtration

This is an aspect of the fact that the action of U^{DR} on $P^{DR}(x)$ is compatible with the Hodge filtration. The action map

$$P^{DR}(x) \times U^{DR} \longrightarrow P^{DR}(x)$$

corresponds to a co-action map

$$\mathcal{A}_x^{DR} \longrightarrow \mathcal{A}_x^{DR} \otimes \mathcal{A}_b^{DR}$$

This is compatible with the Hodge filtration.

The choice of a point $p \in F^0 P^{DR}(x)$ gives an algebra homomorphism $\mathcal{A}_x^{DR} \longrightarrow F$ which kills $F^1 \mathcal{A}_x^{DR}$, which is hence a map of Hodge structures.

De Rham fundamental groups: Hodge filtration

Thus, we get an isomorphism

$$\mathcal{A}_x^{DR} \cong \mathcal{A}_b^{DR}$$

that is compatible with the Hodge filtration. A dimension count then shows that

$$F^1\mathcal{A}_x^{DR} \cong F^1\mathcal{A}_b^{DR},$$

and hence,

$$\mathcal{A}_x^{DR}/F^1\mathcal{A}_x^{DR} \cong \mathcal{A}_b^{DR}/F^1\mathcal{A}_b^{DR},$$

giving us

$$F^0U^{DR} \cong F^0P^{DR}(x).$$

De Rham fundamental groups: crystalline structures

In the local case, the ($k-$linear) Frobenius ϕ of the special fibre Y acts on the category $\mathrm{Un}^{DR}(X)$ [Deligne, Besser].

Write $\mathcal{X} = \cup_i U_i$ so that U_i is a smooth lift of $U_i \otimes k$. Choose local lifts ϕ_i on U_i of the Frobenius on $U_i \otimes k$.

Then given a bundle with connection (V, ∇), we consider the local pull-backs $(\phi_i^*(V|_{U_i}), \phi_I^*(\nabla))$. The connection allows us to patch these together canonically to give us $\phi^*(V, \nabla)$.

In particular,

$$(\mathcal{E}^{DR}, \nabla, 1) \longrightarrow (\phi^*\mathcal{E}^{DR}, \phi^*\nabla, \phi^*1),$$

Get compatible actions on $U^{DR}(V, b)$ and $P^{DR}(X; b, x)$.

De Rham fundamental groups: crystalline structures

On T_n, agrees with the action induced by the isomorphism

$$H^1_{DR}(X) \cong H^1_{crys}(Y).$$

Hence, the eigenvalues are the same as the ones coming from étale cohomology.

Theorem

There is a unique Frobenius invariant element $p^{cr}_{b,x}$ in $P^{DR}(X, b, x)$.

De Rham fundamental groups: crystalline structures

Lemma

The Lang map $L(\phi) : U^{DR} \longrightarrow U^{DR}$ that sends u to $u\phi^{-1}(u)$ is a bijection.

In particular, the identity is the only element fixed by ϕ.

Proof.

The eigenvalues of ϕ on $T_n^{DR} = U^{DR,n}/U^{DR.n+1}$ are all different from 1. □

Proof of theorem.

Choose $p \in P^{DR}$. Then there is a unique $u \in U^{DR}$ such that $\phi(p) = pu$. Write $u = v\phi(v^{-1})$. Then

$$\phi(pv) = pv.$$

Uniqueness comes from the fact that if p is fixed, no pu will be fixed for $u \neq e$. □

De Rham fundamental groups: crystalline structures

Better to think in terms of *crystalline fundamental groups*: Given a point $y \in Y(k)$, define on $\mathrm{Un}(X)^{DR}$ the fibre functor

$$(V, \nabla) \mapsto V(]y[)^{\nabla=0},$$

the flat sections of V over the tube $]y[$ of y, the analytic space of points that reduce to y.

Then for $x, x' \in]y[$, $p^{cr}_{x,x'}$ is given by the diagram

$$\begin{tikzcd} & V(]y[)^{\nabla=0} \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ V_x & & V_{x'} \end{tikzcd}$$

De Rham fundamental groups: crystalline structures

This is supplemented by an isomorphism

$$p^{cr}_{yy'} : V(]y[)^{\nabla=0} \cong V(]y'[)^{\nabla=0}$$

for $y, y' \in Y(k)$ called Coleman integration [Besser]. The computation of this is Kedlaya's theory.

De Rham moduli spaces

The space of torsors for U^{DR} that have compatible Frobenius and Hodge filtration are classified by

$$U^{DR}/F^0.$$

Given a torsor T, choose elements $t^{cr} \in T$ and $t^H \in F^0 T$. Then

$$t^H = t^{cr} u_T^{cr}.$$

The element u_T^{cr} is independent of the choice of t^H up to multiplication by $F^0 U^{DR}$ on the right, giving us a well-defined element

$$[u_T^{cr}] \in U^{DR}/F^0.$$

De Rham fundamental groups

We will give an explicit description for X affine [Kim3].

We first choose

$$\alpha_1, \alpha_2, \cdots, \alpha_m,$$

global algebraic differential forms representing a basis of $H^1_{DR}(X)$.

Thus, $m = 2g + s - 1$, where s is the number of missing points.

De Rham fundamental groups

Consider the algebra

$$F\langle A_1, \ldots, A_m \rangle$$

generated by the symbols $A_1, A_2, \ldots, A_m$. Thus, it is the tensor algebra of the F-vector space generated by the A_i. Let I be the augmentation ideal.

The algebra $F\langle A_1, \ldots, A_m \rangle$ has a natural comultiplication map Δ with values $\Delta(A_i) = A_i \otimes 1 + 1 \otimes A_i$.

Now let

$$E_n = F\langle A_1, \ldots, A_m \rangle / I^{n+1}$$

and take the completion

$$E := \varprojlim F\langle A_1, \ldots, A_m \rangle / I^n$$

Δ extends naturally to a comultiplication $E \rightarrow E \hat{\otimes} E$.

De Rham fundamental groups

$\mathcal{E}$: pro-unipotent pro-vector bundle $E \otimes \mathcal{O}_X$ with the connection ∇ determined by

$$\nabla_{\mathcal{E}} f = df - \Sigma_i A_i f \alpha_i$$

for sections $f : X \longrightarrow E$.

There is an element $1 \in \mathcal{E}_b = E$.

Theorem

There is a unique isomorphism

$$(\mathcal{E}, \nabla_{\mathcal{E}}, 1) \cong (\mathcal{E}^{DR}, \nabla, 1)$$

It is compatible with the comultiplication on either side.

De Rham fundamental groups

The theorem is an easy consequence of

Lemma

Let (V, ∇) be a unipotent bundle with flat connection on X of rank r. Then there exist strictly upper-triangular matrices N_i such that

$$(V, \nabla) \simeq (\mathcal{O}_X^r, d + \Sigma_i \alpha_i N_i)$$

De Rham fundamental groups

The isomorphism

$$\begin{tikzcd}
 & \mathcal{E}^{DR}(]y[)^{\nabla=0} \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
\mathcal{E}_b^{DR} & & \mathcal{E}_x^{DR}
\end{tikzcd}$$

can be constructed locally by solving differential equations.

Let

$$f = \sum_w f_w[w]$$

be a section of $\mathcal{E}$, where the $[w]$ are words in the A_i, and $f(b) = 1$.

Then the flatness condition is

$$df = \sum_w \sum_i f_w \alpha_i [A_i w],$$

De Rham fundamental groups

This is

$$df_{A_i w} = f_w \alpha_i$$

for all w and i.

We solve this iteratively:

$$f_{A_i}(z) = \int_b^z \alpha_i.$$

This can be constructed as a power series with initial condition $f_{A_i}(x) = 0$.

We continue

$$f_{A_j A_i}(z) = \int_b^z f_{A_i} \alpha_j,$$

and so on. Thus, the components of f become iterated integrals.

De Rham fundamental groups

Having solved the equation with initial condition 1, get p_{bx}^{cr} for $v \in \mathcal{E}_b^{DR}$ by

$$p_{bx}^{cr}(v) = f(x)v.$$

For general x, the components of p_{bx}^{cr} give the *definition* of iterated integrals.

The shuffle identities for iterated integrals

$$\int_b^z \alpha_1\alpha_2\cdots\alpha_k \int_b^z \alpha_{k+1}\alpha_{k+2}\cdots\alpha_n = \sum_\sigma \int_b^z \alpha_{\sigma(1)}\alpha_{\sigma(2)}\cdots\alpha_{\sigma(n)}$$

with the sum running over $(k, n-k)$ shuffles of $\{1, 2, \ldots, n\}$ follow from the group-like nature of $p_{b,z}^{cr}$.

De Rham fundamental groups

Another way to say this is that

$$\mathcal{A}_z^{DR} = F[\phi_w]$$

the vector space generated by ϕ_w such that $\phi_w[w'] = \delta_{ww'}$. The algebra structure is given by

$$\phi_w \phi_{w'} = \sum_{\sigma} \phi_{\sigma(ww')},$$

where again the σ run over shuffles. The iterated integral identity is the fact that

$$p_{b,z}^{cr} : \mathcal{A}_z^{DR} \longrightarrow F$$

is an algebra homomorphism.

De Rham fundamental groups

Theorem

The map

$$j^{DR} : X(F) \longrightarrow U^{DR}/F^0$$

has the property that $j^{DR}(]y[)$ *is Zariski dense for each* $y \in Y$.

The idea is to show that all iterated integrals are algebraically independent using transcendental methods.

Hence, as we increase n, the coordinates of the map

$$j^{DR} : X(F) \longrightarrow U_n^{DR}/F^0$$

keep giving genuinely new analytic functions.

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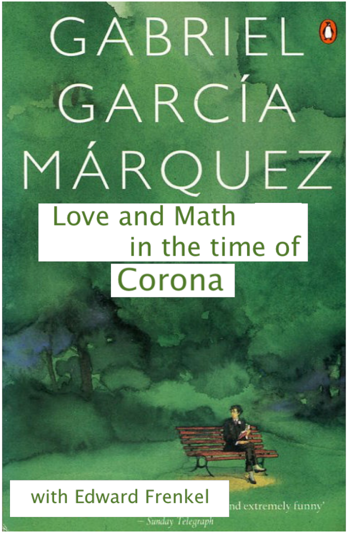

Selmer schemes III

I. Geometry of non-abelian cohomology

Non-abelian cohomology functors

$X/\mathbb{Q}$: a smooth curve and $p > 2$ a place of good reduction.

$U = U(\bar{X}, b)$, the $\mathbb{Q}_p$-prounipotent étale fundamental group.

$U_n = U/U^{n+1}$.

G: either the group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ or $G_T = \mathrm{Gal}(\mathbb{Q}_T/\mathbb{Q})$, where $\mathbb{Q}_T$ is the maximal extension of $\mathbb{Q}$ unramified outside a finite set T of primes. We assume that T contains $\infty, 2, p$ and all primes of bad reduction.

Non-abelian cohomology functors

[Kim1]

We define a functor of $\mathbb{Q}_p$-algebras

$$R \mapsto H^1(G, U_n(R)) := U_n(R) \backslash Z^1(G, U_n(R)).$$

The H^1 refers to continuous cohomology: Z^1 denotes the continuous functions

$$f : G \longrightarrow U(R)$$

such that $f(g_1 g_2) = f(g_1) g_1(f(g_2))$ on which $U_n(R)$ acts via

$$f^u(g) = uf(g)g(u^{-1}).$$

Non-abelian cohomology functors

The G-action on $U_n(R)$ is defined by identifying

$$U_n \cong^{\log} L_n := Lie(U_n).$$

In fact, it is often good to think of U_n as being L_n with group law defined by the BCH formula:

$$X\cdot Y = X+Y+(1/2)[X,Y]+(1/12)[X,[X,Y]]-(1/12)[Y,[Y,X]]+\cdots .$$

(Formula for $\log(\exp(X)\exp(Y))$.)

Then $U_n(R) = L_n \otimes R$.

Non-abelian cohomology functors

The topology in $U_n(R)$ is defined by using

$$U_n \cong \mathbb{A}^N,$$

which gives

$$U_n(R) \cong R^N.$$

We give R^N the inductive limit topology of finite-dimensional $\mathbb{Q}_p$-subspaces. (This definition works also for all affine schemes.)

On the abelian pieces U^n/U^{n+1}, the same definition of H^1 applies, but we can also define H^2.

Proposition

$$H^i(G, U^n/U^{n+1}(R)) \cong H^i(G, U^n(\mathbb{Q}_p)/U^{n+1}(\mathbb{Q}_p)) \otimes R.$$

That is, the functor of R can be represented by the finite-dimensional $\mathbb{Q}_p$-vector space $H^i(G, U^n(\mathbb{Q}_p)/U^{n+1}(\mathbb{Q}_p))$.

Non-abelian cohomology functors

Theorem

The functor

$$R \mapsto H^1(G, U_n(R))$$

is represented by an affine $\mathbb{Q}_p$-scheme of finite type.

The scheme represents principal U_n-bundles with continuous G action:

The R-points are principal $(U_n)_R$ bundles

$$P \longrightarrow \mathrm{Spec}(R),$$

with functorial continuous action of G on $P(S)$ for any R-algebra S.

Non-abelian cohomology functors

The proof is by induction on n using the exact sequence

$$0 \longrightarrow H^1(G, U^n/U^{n+1}(R)) \longrightarrow H^1(G, U_n(R)) \longrightarrow H^1(G, U_{n-1}(R))$$

$$\xrightarrow{\delta} H^2(G, U^n/U^{n+1}(R)).$$

That is, once $H^1(G, U_{n-1})$ is representable, δ is a map of schemes. The exact sequence means that $H^1(G, U_n)$ defines a $H^1(G, U^n/U^{n+1})$-torsor over $\mathrm{Ker}(\delta)$, which then must be represented by

$$\mathrm{Ker}(\delta) \times H^1(G, U^n/U^{n+1}).$$

Non-abelian cohomology functors

In the local case, define also

$$R \mapsto H^1(G, U_n(B_{cris} \otimes R)),$$

and

$$H^1_f(G, U_n) = \mathrm{Ker}(H^1(G, U_n) \longrightarrow H^1(G, U_n(B_{cris})),$$

which is a subscheme by induction on n:

$$\begin{tikzcd}
0 \arrow[r] & H^1(G, U^n/U^{n+1}) \arrow[r] \arrow[d] & H^1(G, U_n) \arrow[r] \arrow[d] & H^1(G, U_{n-1}) \arrow[d] \\
0 \arrow[r] & H^1(G, U^n/U^{n+1}(B_{cris})) \arrow[r] & H^1(G, U_n(B_{cris})) \arrow[r] & H^1(G, U_{n-1}(B_{cris}))
\end{tikzcd}$$

Non-abelian cohomology functors

$H^1_f(G_p, U_n)$ represents torsors that have a G_p-invariant point in $U_n(B_{cris})$.
We have the localisation

$$H^1(G_T, U_n) \longrightarrow H^1(G_p, U_n)$$

using which we define $H^1_f(G_T, U_n) = \mathrm{loc}_p^{-1}(H^1_f(G_p, U_n))$.

Thus, we get a diagram

$$\begin{tikzcd} X(\mathbb{Z}) \arrow[r] \arrow[d] & X(\mathbb{Z}_p) \arrow[d] \\ H^1_f(G_T, U_n) \arrow[r] & H^1_f(G_p, U_n) \end{tikzcd}$$

The bottom arrow is a map of schemes since it represents a map of functors. It is a *computable replacement* for $X(\mathbb{Z}) \subset X(\mathbb{Z}_p)$,

De Rham moduli spaces

The reason $X(\mathbb{Z}_p)$ maps to H^1_f is because of the non-abelian p-adic Hodge theory isomorphism:

$$P_n^{et}(x)(B_{cr}) \cong P_n^{DR}(x)(B_{cr}) \cong B_{cr}^N.$$

The first isomorphism respects all structures, while the second is Galois equivariant, showing the existence of an invariant point.

II. The fundamental diagram

The fundamental diagram

[Kim1, Kim2, Kim3]

Given $T = \mathrm{Spec}(\mathcal{A}(T))$ a crystalline torsor for U,

$$D(T) := \mathrm{Spec}([\mathcal{A}(T) \otimes B_{cr}]^{G_p})$$

is a torsor for U^{DR} with Hodge flitration and Frobenius structure.

Lemma

$$T \mapsto D(T)$$

defines an isomorphism

$$H^1_f(G_p, U_n) \cong U_n^{DR}/F^0.$$

The fundamental diagram

$$\begin{tikzcd}
X(\mathbb{Z}) \arrow[r] \arrow[d] & X(\mathbb{Z}_p) \arrow[d] \arrow[dr] & \\
H^1_f(G_T, U_n) \arrow[r] & H^1_f(G_p, U_n) \cong & U_n^{DR}/F^0
\end{tikzcd}$$

The isomorphism on the right comes from the construction of an inverse using the fundamental exact sequence of p-adic Hodge ttheory:

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_{cr}^{\phi=1} \oplus B_{DR}^{+} \longrightarrow B_{DR} \longrightarrow 0.$$

The fundamental diagram

From this, we get

$$U(B_{DR})/U(B_{DR}^{+}) \longrightarrow H^1(G, U) \longrightarrow H^1(G, U(B_{cr}^{\phi})).$$

For U, we get an equality between

$$H_e^1(G, U) = \mathrm{Ker}[H^1(G, U) \longrightarrow H^1(G, U(B_{cr}^{\phi}))]$$

and

$$H_f^1(G, U) = \mathrm{Ker}[H^1(G, U) \longrightarrow H^1(G, U(B_{cr}))]$$

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