AWS 2020: Geometric quadratic Chabauty

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1 Course description

The quadratic Chabauty method was developed by Kim, Balakrishnan, Besser, Dogra and Müller, and extended in [BDMTV], for finding all rational points on a curve C of genus at least two, provided that $r < g + \rho - 1$. Here, r is the rank of $J(\mathbb{Q})$, with J the jacobian of C, g is the genus of C, and ρ is the Picard number (over $\mathbb{Q}$) of J.

The course has two aims. To describe the quadratic Chabauty method in terms of algebraic geometry only: models over the integers of line bundles on J. And to give an algorithm that can verify, in each given instance where $r < g + \rho - 1$, that the list of known rational points is complete. The course does not aim at effective or uniform finiteness results for *classes of curves.*

The course will follow the preprint [E-L], providing more background where or when needed. The number $g + \rho - 1$ is the dimension of a product T of $\rho - 1$ principal $\mathbb{G}_{\mathrm{m}}$-bundles on J. As in the classical (linear) Chabauty method, we are intersecting, for p a prime number, but now in $T(\mathbb{Q}_p)$ in stead of in $J(\mathbb{Q}_p)$, the closure of $T(\mathbb{Z})$, which has dimension $\leq r$, with $C(\mathbb{Q}_p)$.

Planning of the lectures

The following planning is preliminary, and will be adapted as the course goes on. One idea is to carry the example in Section 8 along through all the lectures.

1. Section 2.
2. Sections 3 and 4.
3. Section 5.
4. Sections 6 and 7 (not in detail).
5. ?

2 Projects and required background

2.1 Translation from the geometric approach to the fundamental group theoretical approach

The aim of this project is to relate the fundamental group approach to quadratic Chabauty as in [BDMTV] to the geometric method in [E-L]. It is claimed that experts more or less know how to do this. It is also true that the authors of the articles just mentioned do *not* know the details, but are really interested in them.

Our starting point is the geometric approach as in [E-L], and, more precisely, the diagram (2.12) on page 5 (see there for details):

$$\begin{tikzcd}
 & T \arrow[r] \arrow[d] & P^{\times,\rho-1} \arrow[d] \\
U \arrow[ru, "\widetilde{j_b}"] \arrow[r, "j_b"] & J \arrow[r, "(\mathrm{id}, m\cdot\circ\mathrm{tr}_{c_i}\circ f_i)_i"] & J \times (J^{\vee 0})^{\rho-1} \,.
\end{tikzcd}$$

In [M-B], Theorem 5.4, Moret-Bailly shows how P is naturally equipped with metrics on its fibres over $J(\mathbb{C}) \times J^{\vee}(\mathbb{C})$, and that for (x, y) in $(J \times J^{\vee 0})(\mathbb{Z})$, the Arakelov degree of $(x, y)^*P$ is the Néron-Tate height of (x, y).

Now T is the product of $\rho - 1$ principal $\mathbb{G}_{\mathrm{m}}$-bundles T_i on J. Their associated line bundles $\mathcal{L}_i$ are pullbacks of P and so have natural metrics over $J(\mathbb{C})$. The pullbacks of the $\mathcal{L}_i$ to U are trivial (uniquely up to signs) and the norms of the trivialising sections are constant as functions on $U(\mathbb{C})$, because they are harmonic. These constants $c_{U,i}$ can be read off from (7.8) and (7.2) in [E-L]. For any u in $U(\mathbb{Z})$, the heights of $j_b(u)$ with respect to $\mathcal{L}_i$ are equal to $-\log(c_{U,i})$. Note that the heights on $J(\mathbb{Q})$ attached to the $\mathcal{L}_i$ are not necessarily positive because the $\mathcal{L}_i$ are not ample on $J_{\mathbb{Q}}$ (they are trivial on $C_{\mathbb{Q}}$). These heights are $\mathbb{R}$-valued polynomial functions of degree at most 2 on $J(\mathbb{Q})$, the coefficients of which (with respect to a $\mathbb{Z}$-basis of $J(\mathbb{Q})$ modulo torsion) are given by the corresponding values of the Néron-Tate height pairing. The knowledge of the heights of the $j_b(u)$ with respect to the $\mathcal{L}_i$ could be very useful information for finding out which elements of $J(\mathbb{Q})$ are in $C(\mathbb{Q})$, but the non-definiteness of the quadratic polynomials and the complexity of their coefficients make it harder. Nevertheless, looking further into this may be an interesting project, especially for modular curves, where ρ is equal to g, and where there are strong results on the Birch and Swinnerton-Dyer conjecture for the isogeny factors of the jacobian where the analytic rank of the L-function is at most 1.

So, following Chabauty, one tries a p-adic approach. Here this means considering, for some chosen prime p, p-adic valued heights and Arakelov theory. This is done in [M-T] for abelian varieties, using biextensions, and in [C-G] for jacobians. The idea is that the product formula must be preserved and that the analytic functions at the archimedean places are replaced by p-adic analytic functions at the p-adic places. For example, the $\mathbb{R}$-valued adele norm on the ideles of $\mathbb{Q}$, $x \mapsto \|x\| = \prod_v |x_v|_v$, is trivial on $\mathbb{Q}^{\times}$. For $x \in \mathbb{R}^{\times}$ we have $|x|_{\infty} = x\cdot\mathrm{sign}(x)$. The factor x can be moved, for $x \in \mathbb{Q}^{\times}$, to any p-adic place of our choice. So, for a prime p, we obtain the $\mathbb{Q}_p^{\times}$-valued adele norm $x \mapsto (x_p\cdot|x_p|_p)\cdot\mathrm{sign}(x_{\infty})\cdot\prod_{v\notin\{p,\infty\}} |x_v|_v$, also trivial on $\mathbb{Q}^{\times}$. Up to a sign, it corresponds via class field theory for $\mathbb{Q}$ to the p-adic cyclotomic character. The main point is that at all places other than p and ∞, nothing has changed.

We are now already very close to § 1.4 of [BDMTV], and it should not be very hard to get a precise translation.

For the subsequent interpretation in [BDMTV] of everything in terms of fundamental groups, we note that the embedding $\widetilde{j_b}\colon C_{\mathbb{Q}} \to T_{\mathbb{Q}}$ induces a morphism of fundamental groups. The complex uniformisation of $P^{\times}(\mathbb{C})$ (see [B-E, §4]) gives the structure of $\pi_1(P^{\times}(\mathbb{C}))$; it is a non-abelian extension of $\pi_1(J(\mathbb{C}) \times J(\mathbb{C}))$ by $\mathbb{Z}$. So, apparently, one has to study p-adic local systems on $T_{\mathbb{Q}}$.

Required background.

Basic knowledge of the algebraic geometry in [E-L], mainly over $\mathbb{C}$ and over $\mathbb{Q}$. Some Arakelov height theory (see [M-B], [H-S], and [H]) and p-adic height theory ([M-T] and [C-G])).

For the passage from p-adic heights to fundamental groups, some working knowledge of Galois cohomology and etale cohomology (see [Po]), algebraic de Rham cohomology (see `https://en.wikipedia.org/wiki/Khler_differential`), knowledge in abelian and non-abelian p-adic Hodge theory (see the references in [BDMTV]).

2.2 Comparing computations with participants to Jennifer Balakrishnan's project 2: modular curves $X_0(n)^+$.

The aim here is to apply the geometric quadratic Chabauty method to the curves $X_0(n)^+$ mentioned in Jennifer Balakrishnan's project, and then to compare the whole process with the participants of that project.

We hope that this comparison gives some insight in running times on both sides, actually even for linear Chabauty (as treated in David Zureick-Brown's lectures): Coleman integrals on $C(\mathbb{Q}_p)$ versus computations in $J(\mathbb{Z}/p^2\mathbb{Z})$.

Here one can build on Guido Lido's example (Section 8 in [E-L]) and his code in cocalc, to be found with [E-L]. It may be that at the time of the School the example $X_0(73)^+$ will be available.

Required background

Section 8 (and therefore most of the other sections as well) of [E-L]. Some knowledge of modular curves, see [D-S].

2.3 Generalisation of the geometric quadratic Chabauty method to number fields

This generalisation has already been carried out for bielliptic curves of genus 2 in [BBBM]. It is interesting to see, at first theoretically, how the methods of [E-L] can be generalised to number fields. The first idea is to use Weil restriction, to reduce to geometry over $\mathbb{Q}$.

Required background

Section 2 of [E-L].

References

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