AWS 2020: Geometric quadratic Chabauty

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1 Course outline

The quadratic Chabauty method was developed by Balakrishnan and Dogra and extended in [BDMTV], for finding all rational points on a curve C of genus at least two, provided that $r < g + \rho - 1$. Here, r is the rank of $J(\mathbb{Q})$, with J the jacobian of C, g is the genus of C, and ρ is the Picard number (over $\mathbb{Q}$) of J.

The course has two aims. To describe the quadratic Chabauty method in terms of algebraic geometry only: models over the integers of line bundles on J. And to give an algorithm that can verify, in each given instance where $r < g + \rho - 1$, that the list of known rational points is complete. The course does not aim at effective or uniform finiteness results for *classes of curves.*

The course will follow the preprint [E-L], providing more background where or when needed. The number $g + \rho - 1$ is the dimension of a product T of $\rho - 1$ principal $\mathbb{G}_{\mathrm{m}}$-bundles on J. As in the classical (linear) Chabauty method, we are intersecting, for p a prime number, but now in $T(\mathbb{Q}_p)$ in stead of in $J(\mathbb{Q}_p)$, the closure of $T(\mathbb{Z})$, which has dimension $\leq r$, with $C(\mathbb{Q}_p)$.

2 Projects and required background

1. Translation between the fundamental group approach and the geometric approach to quadratic Chabauty.

   The aim is to understand how to pass from the geometric description to the one in [BDMTV]. In particular: how do p-heights come in? The freshness of this question and the presence of practicioners of both sides make the Arizona Winter School an ideal opportunity for this. The starting point is the fundamental group of $T(\mathbb{C})$ (see [B-E, §4]), and then p-adic local systems on $T_{\mathbb{Q}}$.

   Required background. Basic knowledge of the algebraic geometry in [E-L], mainly over $\mathbb{C}$ and over $\mathbb{Q}$. Some height theory (Arakelov and p-adic). Some working knowledge of Galois cohomology, etale cohomology, and algebraic de Rham cohomology. References: [Po], [H-S], and [H].

2. Comparing computations with participants to Jennifer Balakrishnan's project 2: modular curves $X_0(n)^+$.

   The aim here is to apply the geometric quadratic Chabauty method to the curves $X_0(n)^+$ mentioned in Jennifer Balakrishnan's project, and then to compare the whole process with the participants of that project.

   We hope that this comparison gives some insight in running times on both sides, actually even for linear Chabauty (as treated in David Zureick-Brown lectures): Coleman integrals on $C(\mathbb{Q}_p)$ versus computations in $J(\mathbb{Z}/p^2\mathbb{Z})$.

   Here one can build on Guido Lido's code in cocalc, to be found with [E-L].

   Required background: Section 8 of [E-L].

3. Generalise the geometric quadratic Chabauty method from $\mathbb{Q}$ to number fields, and compare with [BBBM].

   The idea is to use Weil restriction, to reduce to geometry over $\mathbb{Q}$.

   Required background: Section 2 of [E-L].

References

[BBBM] Jennifer S. Balakrishnan, Amnon Besser, Francesca Bianchi, Steffen Müller. *Explicit quadratic Chabauty over number fields.*
https://arxiv.org/pdf/1910.04653v1.pdf

[BDMTV] Jennifer Balakrishnan, Netan Dogra, Steffen Müller, Jan Tuitman, Jan Vonk. *Explicit Chabauty-Kim for the split Cartan modular curve of level 13.* Ann. of Math. (2) 189 (2019), no. 3, 885–944.
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[B-E] Daniel Bertrand and Bas Edixhoven, *Pink's conjecture on unlikely intersections and families of semi-abelian varieties.*
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[E-L] Bas Edixhoven and Guido Lido. *Geometric quadratic Chabauty.*
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[H-S] Marc Hindry and Joseph Silverman. *Diophantine geometry. An introduction.* Graduate Texts in Mathematics, 201. Springer-Verlag, New York, 2000.

[H] Johan Huisman. *Heights on abelian varieties.* Chapter 5 in "Diophantine approximation and abelian varieties. Introductory lectures." Papers from the conference held in Soesterberg, April 12–16, 1992. Edited by Bas Edixhoven and Jan-Hendrik Evertse. Lecture Notes in Mathematics, 1566. Springer-Verlag, Berlin, 1993.
http://pub.math.leidenuniv.nl/~edixhovensj/publications/1993/Edixhoven-Evertse_Soesterberg_1992.pdf

[Po] Bjorn Poonen. *Rational points on varieties.* Graduate Studies in Mathematics, 186. American Mathematical Society, Providence, RI, 2017.
http://www-math.mit.edu/~poonen/papers/Qpoints.pdf