Lecture 4.

① $$\begin{tikzcd} \mathbb{Z}_p^r \arrow[r, "\kappa"] & T(\mathbb{Z}_p)_t \\ & U(\mathbb{Z}_p)_u \arrow[u, "\tilde{j}_b"'] \end{tikzcd}$$

$r \leq g+\rho-2$.

Question: how can we prove, without doing computations, that $\kappa(\mathbb{Z}_p^r) \cap U(\mathbb{Z}_p)_u$ is finite? This should follow from our choice of $\mathcal{L}_1, \ldots, \mathcal{L}_{\rho-1}$ on J, namely that they are $\mathbb{Z}$-lin. ind. in $NS(J_{\mathbb{Q}})$.

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② $A^r_{\mathbb{F}_p} \xrightarrow{\bar{\kappa}} T_{T_{\mathbb{F}_p}}(t)$ can be non-injective,

for example if $\rho=1$, $J=T$, and

$$J(\mathbb{Z})_{0_{\mathbb{F}_p}} = \ker\left(J(\mathbb{Z}) \to J(\mathbb{Z}/p^2\mathbb{Z})\right). \ (\text{then } \bar{\kappa}=0).$$

Two aims. ① Show that A as in Thm 4.12 can be computed, i.e. make it explicit.

② Show that the computations are not so bad.

①: § 6–7.

§6. "Rigidify line bundles on C at b".

$$\mathrm{Pic}_{C/S}(T) = \frac{\mathrm{Pic}(C_T)}{\mathrm{Pic}(T)} = \left\{ (\mathcal{L}, \varphi) : \begin{array}{l} \mathcal{L} \text{ inv. } \mathcal{O}\text{-module on } C_T \\ \varphi : \mathcal{O}_T \xrightarrow{\sim} b_T^* \mathcal{L} \end{array} \right\} \Big/ \cong$$

C $\quad \mathbb{Z}[1/n]$

$T \to S$

such objects have no nontrivial automorphisms.

<u>Prop. 6.3.2</u> S any scheme, $C \to S$ proper smooth curve, of genus $g \geq 1$, $b \in C(S)$, $d \in \mathbb{Z}_{\geq 0}$.

$\exists!$ morphisms of $\mathbb{G}_m$-torsors, compatible with rigidif.'ns:

$$
\begin{tikzcd}
\mathcal{L}^{\mathrm{univ},\times} \arrow[r] \arrow[d] & P^\times \arrow[d] & \mathcal{M}^\times \arrow[l] \arrow[d] & \mathcal{N}_d^\times \arrow[l] \arrow[d] & ?? \arrow[l] \\
C \times_S J \arrow[r, "j_b \times j_b^{*,-1}"'] & J \times_S J^\vee & J \times_S J \arrow[l, "\sim", "\mathrm{id} \times j_b^{*,-1}"'] & J \times_S C^{(d)} \arrow[l, "\mathrm{id} \times \Sigma"] & \\
 & & & C^d/S_d \arrow[u] &
\end{tikzcd}
$$

mod. space of eff. div. degree d on C/S. $\to C^d/S_d$

with: $\mathcal{N}_d := \mathrm{Norm}_{D_J^{\mathrm{univ}}/J \times_S C^{(d)}} \left(\mathcal{L}^{\mathrm{univ}}_{C^{(d)}} \right)$

$$\mathcal{L}^{\mathrm{univ}}_{C^{(d)}} \to C \times_S J \times_S C^{(d)}$$

where:

$$
\begin{tikzcd}
D^{\mathrm{univ}} \arrow[r, hook] \arrow[dr, "\text{finite loc. free of degree } d"'] & C \times C^{(d)} \arrow[d] \\
 & C^{(d)}
\end{tikzcd}
\qquad
\begin{tikzcd}
D_J^{\mathrm{univ}} \arrow[r, hook] & C \times_S J \times_S C^{(d)} \arrow[d] \\
 & J \times_S C^{(d)}
\end{tikzcd}
$$

For example (6.3.12) $\quad x_i, y_i, u_j, v_j \in C(S)$

$$\mathcal{M}\Big(\sum_i (x_i - y_i), \sum_j (u_j - v_j)\Big) =$$

$$= \bigotimes_j \Big(u_j^* \mathcal{O}_C\big(\sum_i (x_i - y_i)\big) \otimes_{\mathcal{O}_S} v_j^* \mathcal{O}_C\big(\sum_i (y_i - x_i)\big)\Big)$$

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§7. $\alpha := tr_{b_i} \circ f_i : J_{\mathbb{Q}} \to J_{\mathbb{Q}}$; <u>make explicit.</u>

$$\begin{tikzcd}
 & T_\alpha \arrow[rr] \arrow[d] & & \mathcal{M}^\times \arrow[dl] \\
C_{\mathbb{Q}} \arrow[r, "jb"] \arrow[d, "diag"] & J_{\mathbb{Q}} \arrow[r, "(id,\alpha)"] & (J\times J)_{\mathbb{Q}} & \\
(C\times C)_{\mathbb{Q}} \arrow[r, "id\times jb"'] & (C\times J)_{\mathbb{Q}} \arrow[r, "id\times\alpha"'] & (C\times J)_{\mathbb{Q}} \arrow[u, "jb\times id"'] & \\
\mathcal{L}_\alpha^\times \arrow[u] \arrow[rrr] & & & \mathcal{L}^{univ,\times} \arrow[ul] \arrow[uuu]
\end{tikzcd}$$

$jb^* T_\alpha$ trivial on $C_{\mathbb{Q}}$ $\iff$ $\mathcal{L}_\alpha$ trivial on diagonal in $(C\times C)_{\mathbb{Q}}$.

Computations in $T(\mathbb{Z}/p^2\mathbb{Z})$, $J(\mathbb{Z}/p^2\mathbb{Z})$, $J(\mathbb{F}_p)$.

1. $J(\mathbb{Z}/p^2\mathbb{Z})$, $J(\mathbb{F}_p)$: line bundles up to isom. on $C_{\mathbb{Z}/p^2\mathbb{Z}}$ or $C_{\mathbb{F}_p}$, e.g. work with divisors.

2. $T(\mathbb{Z}/p^2\mathbb{Z})$: no extra effort if you use rigidified line bundles.

Finally we get back to J, good or bad?

Compare with $Gr_{2g,g}$ Grassmannian of g-dim. subsp. in a $2g$ dim. space.

dim: / g^2.

embedded in: $\mathbb{P}^{\binom{2g}{g}-1}$ $\binom{2g}{g} \geq \frac{2^{2g}}{2g+1}$.

Not a problem: just describe subspaces by giving a basis.

Now J. Just the same.

Fix $\mathcal{L}$ on C of degree $\geq 3g-1$.

Then $\forall\ D \geq 0$ eff. div. degree g

$$H^0(C, \mathcal{L}(-D)) \hookrightarrow H^0(C, \mathcal{L})$$

is a codim. g subspace.

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