Lecture 3: §3 & Thm. 4.12.

§3. p any prime number.

X smooth $\mathbb{Z}_p$-scheme, rel. dim. d

$$x \in X(\mathbb{F}_p) \qquad \begin{array}{ccc} X(\mathbb{Z}_p) & \longrightarrow & X(\mathbb{F}_p) \\ \cup & & \cup \\ X(\mathbb{Z}_p)_x & \longrightarrow & \{x\} \end{array}$$

Let $p, t_1, \ldots, t_d$ gen. of max. id. in $\mathcal{O}_{X,x}$.

$$\begin{tikzcd} X(\mathbb{Z}_p)_x \arrow[r, "t_1,\ldots,t_d", "\sim"'] \arrow[rr, bend right] & p\mathbb{Z}_p^d \arrow[r, "\frac{1}{p}\cdot", "\sim"'] & \mathbb{Z}_p^d \end{tikzcd}$$

$$\tilde{t} := (\tilde{t}_1, \ldots, \tilde{t}_d) = \left(\frac{t_1}{p}, \frac{t_2}{p}, \ldots, \frac{t_d}{p}\right)$$

Geometry. shrink X, s.t. it is affine, t_i regular

and

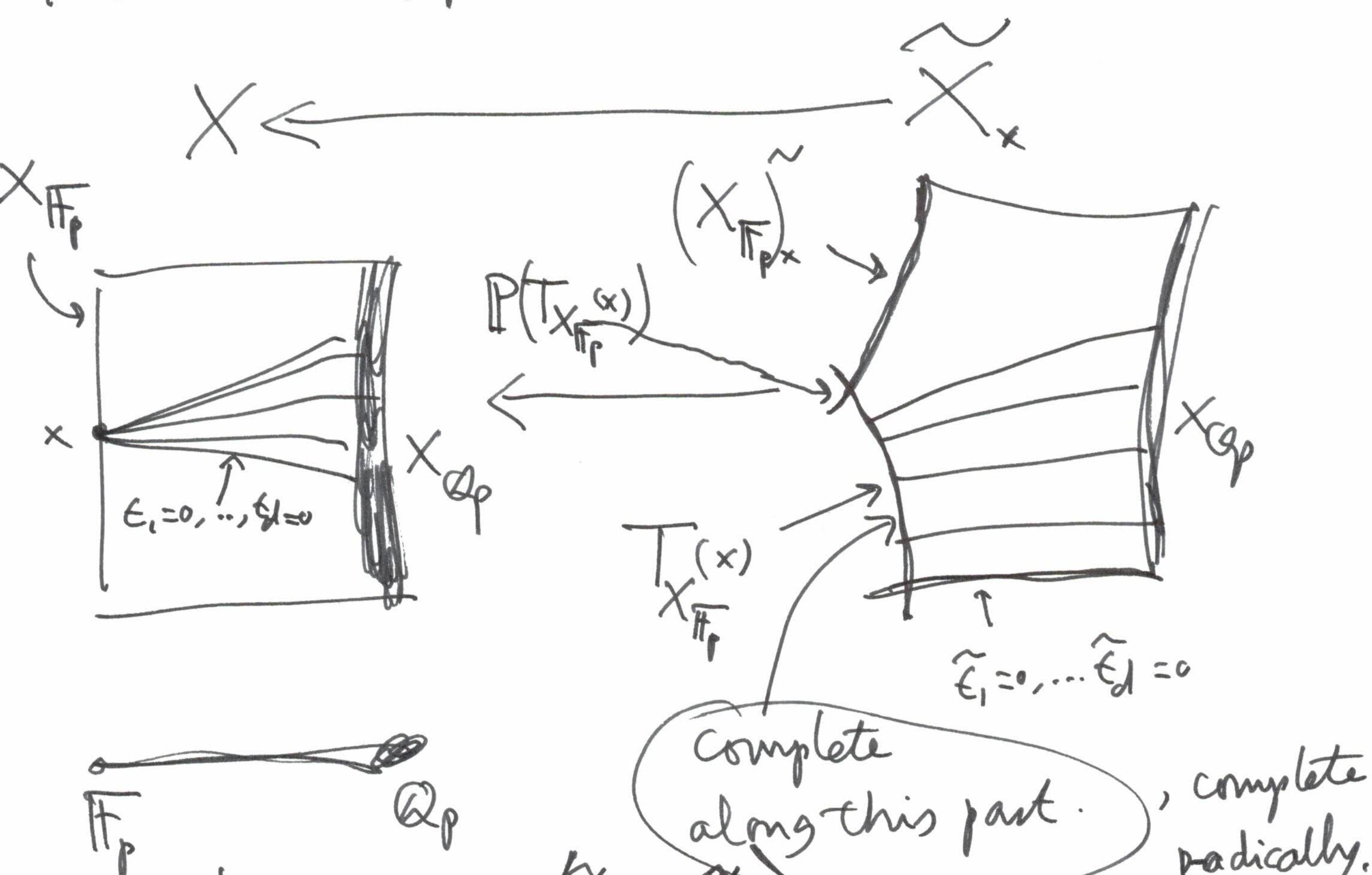
$$\begin{tikzcd}
X \arrow[r, "t", "\text{etale}"'] & \mathbb{A}^d_{\mathbb{Z}_p} & \mathbb{Z}_p[t_1,\dots,t_d] \arrow[d] & t_i \arrow[d, mapsto] \\
\tilde{X}_x \arrow[u, "\pi"] \arrow[r] & \tilde{\mathbb{A}}^d_{\mathbb{Z}_p,0} \arrow[u] & \mathbb{Z}_p[\tilde t_1,\dots,\tilde t_d] & p\cdot\tilde t_i \\
\tilde{X}^{pL}_x \arrow[u, phantom, "\cup" description] \arrow[r] & \mathbb{A}^d_{\mathbb{Z}_p} \arrow[u] \arrow[uu, bend right, "\cdot p"'] & &
\end{tikzcd}$$

$t^{-1}\{0_{\mathbb{F}_p}\} = \{x\}$.

blow up in x

$\tilde t_i = \frac{t_i}{p}$.

Picture ("$d=1$, $p=5$")



This gives: $\mathbb{Z}_p\langle\tilde t_1,\dots,\tilde t_d\rangle = \mathbb{Z}_p[\tilde t_1,\dots,\tilde t_d]^{\wedge p}$

$\mathcal{O}(\tilde{X}^p_x)^{\wedge p} = \mathbb{Z}_p\langle\tilde t_1,\dots,\tilde t_d\rangle$

$$\mathbb{F}_p \otimes_{\mathbb{Z}_p} \left(\mathcal{O}\left(\widetilde{X}^p_x\right)^{\wedge_p}\right) = \mathbb{F}_p[\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_d].$$

geom: $T_{X_{\mathbb{F}_p}}(x)$. $\qquad p > 2$.

here we want $p \nmid n$ (good red. at p)

On Theorem 4.12.

Situation:

(affine) linear embedding of codim. $g+\rho-2$

tangent map of $\tilde{j}_b$ at u.

$$\begin{tikzcd}
T_{U_{\mathbb{F}_p}}(u) = (\widetilde{U}^p_u)_{\mathbb{F}_p} \arrow[r] \arrow[d, hook] & (\widetilde{T}^p_t)_{\mathbb{F}_p} \arrow[d] \\
\widetilde{U}^p_u \arrow[r] \arrow[d] & \widetilde{T}^p_t \arrow[d] \\
U \arrow[r, "\tilde{j}_b"] & T \\
U(\mathbb{F}_p) \ni u \arrow[r, mapsto] & t
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \mathbb{F}_p \otimes \mathcal{O}(\widetilde{T}^p_t)^{\wedge_p} \ni \bar{f}_1, \dots, \bar{f}_{g+\rho-2} & \\
 & & & (f_1, \dots, f_{g+\rho-2}) \arrow[dl, hook] \arrow[dd, "(k^* f_i)_i ="] \\
\mathcal{O}(\widetilde{U}^p_u)^{\wedge_p} & & \mathcal{O}(\widetilde{T}^p_t)^{\wedge_p} \arrow[ll, two heads] \arrow[uu] \arrow[d, "k^*"] & \\
A \arrow[d, two heads] & & \mathbb{Z}_p\langle z_1, \dots, z_r \rangle \arrow[ll, two heads] \arrow[d, two heads] & I \arrow[l, hook'] \arrow[d] \\
\bar{A} & & \mathbb{F}_p[z_1, \dots, z_r] \arrow[ll, two heads] & \bar{I} \arrow[l, hook']
\end{tikzcd}$$

Thm. 4.12: $\overline{f_1}, \ldots, \overline{f_{g+p-2}}$: deg $\leqq 1$.

~~$\kappa^* \overline{f_1}, \ldots, \kappa^* f$~~

$f_1, \ldots, f_{g-1} \in \mathcal{O}_{J, j_b(u)}$

$\kappa^* \overline{f_1}, \ldots, \kappa^* \overline{f_{g-1}}$: deg ≤ 1.

$\kappa^* \overline{f_g}, \ldots, \kappa^* f_{g+p-2}$: deg ≤ 2.

One can compute the $\kappa^* \overline{f_i}$ in terms of $\mathbb{F}_p^r \xrightarrow{\overline{\kappa}} T(\mathbb{Z}/p^2\mathbb{Z})$

If $\overline{A}$ is finite, then $\dim_{\mathbb{F}_p}(\overline{A}) \geqslant \# U(\mathbb{Z})_u$.

Proof: ~~A~~ A is p-adically complete. $\boxtimes$

A is a f.g. $\mathbb{Z}_p$-module

Hence $\operatorname{rank}_{\mathbb{Z}_p}(A_{red}) \leqslant \dim_{\mathbb{F}_p}(\overline{A})$

$\leqslant$

$\# \mathrm{Hom}_{\mathbb{Z}_p}(A_{red}, \mathbb{Z}_p)$

$\shortparallel$

$\# (U(\mathbb{Z})_u)$