Geometric Quadratic Chabauty, $\frac{0}{1}$.

(jt. work with Guido Lido).

C nice curve $/\mathbb{Q}$, assume $b \in C(\mathbb{Q})$.
$g > 1$.

$j_b : C \longrightarrow J$, $P \mapsto \mathcal{O}_C(P-b)$.

Chabauty has problem if $r \nless g$.

$$\begin{tikzcd}
C(\mathbb{Q}) \arrow[r] \arrow[dddd] & J(\mathbb{Q}) \\
& \cap \\
& \overline{J(\mathbb{Q})} & \leftarrow \text{can be equal.} \\
& \cap \\
C(\mathbb{Q}_p) \arrow[r] & J(\mathbb{Q}_p)
\end{tikzcd}$$

Idea: replace J by something bigger, higher dimension, and then play Chab game.

What to take? A $\mathbb{G}_m$-torsor on J.

line bundle minus 0-section

$$\begin{tikzcd}
& j_b : C \arrow[r] & J & & \\
& & \ker(j_b^*) \arrow[r, "\sim"] \arrow[d] & \ker(j_{b,NS}^*) \arrow[d] & \\
\textcolor{red}{J^\vee(\mathbb{Q})} \quad J^\vee \arrow[r, hook] \arrow[d, "-\lambda^{-1} = j_b^*"'] & & \mathrm{Pic}_{J/\mathbb{Q}} \arrow[r, two heads] \arrow[d, "j_b^*"] & NS_{J/\mathbb{Q}} = \underline{\mathrm{Hom}}(J, J^\vee)^+_{\mathbb{Q}} \arrow[d, "j_b^*"] & \\
J \arrow[r, hook] & & \mathrm{Pic}_{C/\mathbb{Q}} \arrow[r, two heads, "\deg"] & \mathbb{Z}_{\mathbb{Q}} &
\end{tikzcd}$$

$\textcolor{red}{f \mapsto (id, f)^* P}$

$\ker(j_{b,NS}^*) \leftarrow$ rank $\rho - 1$

trace on $H_1(J(\mathbb{C}), \mathbb{Z})$

Free $\mathbb{Z}$-module of rank ρ.

Given $f : J \to J^\vee$ with trace 0, get a line bundle on J that is trivial on $j_b(C)$.

Poincaré bundle.

$\forall$ ab. var. A, can view $A^\vee$ as $\mathrm{Ext}^1(A, \mathbb{G}_m)$. $\mathbb{G}_m \hookrightarrow E \twoheadrightarrow A$

$\mathbb{G}_{m,A}$-torsor.

are rigid. no non-trivial autom. that induce id_A and $\mathrm{id}_{\mathbb{G}_m}$.

So over $A^\vee$ have univ. extension:

$$\begin{tikzcd}
\mathbb{G}_{m,A^\vee} \arrow[r, hook] \arrow[dr] & P^\times \arrow[r, two heads] \arrow[d] & A\times A^\vee \arrow[dl] \\
& A^\vee &
\end{tikzcd}$$

$P^\times$ is also the univ. ext. of $A^\vee$ by $\mathbb{G}_m$, over A.

$P^\times$ is a $\mathbb{G}_m$-biextension of $A\times A^\vee$:

2 partial gr. laws: $z_1 +_1 z_2 \in P^\times(x_1+x_2, y)$

$x_1, x_2 \in A$, $y \in A^\vee$

$(x_1, y) \in A\times A'$ $\quad z_1 \in P^\times(x_1, y)$

$(x_2, y) \in$ $\quad z_2 \in P^\times(x_2, y)$

$$\begin{tikzcd}
j_b^* T_f \arrow[r] \arrow[d] & T_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & P^\times \arrow[d] & \\
C \arrow[ur, "\tilde{j}_b"] \arrow[r, "j_b"'] & J \arrow[r, "{(id,\, tr_{b_f} \circ f)}"'] & J \times J^\vee & b_f.
\end{tikzcd}$$

T_f: dim. $g+1$. $\qquad J$: dim g.

Take a $\mathbb{Z}$-basis $f_1, \dots, f_{\rho-1}$ of $\ker(j_{b\,NS}^*)$,

gives

$$\begin{tikzcd}
 & T \arrow[r] \arrow[d] & (P^\times)^{\rho-1} \arrow[d] \\
C \arrow[ur, "\tilde{j}_b"] \arrow[r, "j_b"'] & J \arrow[r, "{(id,\, tr_{b_{f_1}} \circ f_1, \dots, tr_{b_{f_{\rho-1}}} \circ f_{\rho-1})}"'] & J \times (J^\vee)^{\rho-1}
\end{tikzcd}$$

T: dim. $g+\rho-1$. $\qquad (P^\times)^{\rho-1}$: $\mathbb{G}_m^{\rho-1}$-torsor.

We play the Chabauty game in T.

Hope that it works if $r < g+\rho-1$.

(Most wanted ex. have $\rho = g$.)

$T(\mathbb{Q})$ is a $\mathbb{Q}^{\times,\rho-1}$-torsor.

$\downarrow$

$\mathbb{Q}^\times = \{\pm 1\} \times \mathbb{Z}^{(\text{set of primes})}$

$J(\mathbb{Q})$

Big problem, too many $\mathbb{Q}$-points in T.

<u>Solution</u>: extend the geometry over $\mathbb{Z}$. $\mathbb{Z}^\times = \{\pm 1\}$.

<u>From now on everything over $\mathbb{Z}$.</u>

C proper regular model of $C_\mathbb{Q}$.

$J :=$ Néron model of $J_\mathbb{Q}$.

$J^\vee :=$ ——— $J^\vee_\mathbb{Q}$, $J^{\vee,0} \subset J^\vee$: fibrewise conn. comp. of 0.

$P^\times$: unique extension of $P^\times_\mathbb{Q}$ to $J \times J^{\vee,0}$ as <u>bi-extension</u>.

We use the biext. str. of $P^\times$ over $J \times J^{\vee,0}$ to parametrize.

$$\begin{array}{l} T(\mathbb{Z})_{(\pm 1)^{p-1}} \\ \big\downarrow \\ J(\mathbb{Z}) \end{array}$$

torsor.

$$\begin{array}{ccc} C(\mathbb{Z}) & \longrightarrow & T(\mathbb{Z}) \\ \big\downarrow & & \cap \\ & & \overline{T(\mathbb{Z})} \\ & & \cap \\ C(\mathbb{Z}_p) & \longrightarrow & T(\mathbb{Z}_p) \end{array}$$

$\overline{T(\mathbb{Z})}$: $\leq r$-dim.

$C(\mathbb{Z}_p)$: 1-dim.

$T(\mathbb{Z}_p)$: $g+p-1$-dim.