Our set-up:

$X/\mathbb{Q}$ nice curve of genus $g \geq 2$

$\operatorname{rk} J(\mathbb{Q}) = g$, $\operatorname{rk} NS(J) > 1$, $\log : J(\mathbb{Q}) \otimes \mathbb{Q}_p \xrightarrow{\sim} H^0(X_{\mathbb{Q}_p}, \Omega^1)^*$

p good prime

X has everywhere pot. good reduction.

We extend yesterday's diagram:

$$\begin{tikzcd}
X(\mathbb{Q}) \arrow[r] \arrow[d, "j"] & X(\mathbb{Q}_p) \arrow[d, "j_p"] \arrow[dr, "\text{iterated Coleman integrals}"] & \\
H^1_f(G_T, U) \arrow[r, "\mathrm{loc}_p"] \arrow[d, "\tau"] & H^1_f(G_p, U) \arrow[r] \arrow[d, "\tau_p"] & U^{dR}/\mathrm{Fil}^0 \\
M_{\mathbb{Q},f} \arrow[r] \arrow[d] & M_p \arrow[r] & M_{\mathrm{fil},\phi} \arrow[ddl, bend left, "h_p"] \\
H^1_f(G_T, V) \times H^1_f(G_T, V^*(1)) \arrow[dr, "h"] & & \\
& \mathbb{Q}_p &
\end{tikzcd}$$

Construct τ, τ_p via twisting.

Then by our assumptions, have the following:

Nekovar ht : $H^1_f(G_T, V) \times H^1_f(G_T, V^*(1))$

$\quad \wr\wr$ (loc$_p$, Poincaré duality)

$H^1_f(G_p, V) \times H^1_f(G_p, V)$

$\quad \wr\wr$ (Bloch-Kato log)

$H^0(X, \Omega^1)^* \times H^0(X, \Omega^1)^*$

so we may view Nekovář height as a bilinear pairing

Now let $A : X(\mathbb{Q}) \longrightarrow M_{\mathbb{Q}, f}$

$x \longmapsto \tau(j(x))$

and do this similarly for $x \in X(\mathbb{Q}_p)$

$\Rightarrow x \mapsto h(\tau(j(x)))$ extends to

$X(\mathbb{Q}_p) \rightarrow \mathbb{Q}_p$

Fix a basis $\{\Psi_i\}$ of $H^0(X, \Omega^1)^* \otimes H^0(X, \Omega^1)^*$
rewrite ht in terms of this basis, using known $\mathbb{Q}$-points (either if enough $X(\mathbb{Q})$ or $J(\mathbb{Q})$)

Thm (B-Dogra) QC for rational points

The function $\rho : X(\mathbb{Q}_p) \to \mathbb{Q}_p$

$$x \mapsto h_p(A(x)) - h(A(x))$$

vanishes on $X(\mathbb{Q}_p)_u$ and has finitely many zeros.

To make this explicit, need to

1) Write h in terms of basis of $H^0(X,\Omega^1)^* \otimes H^0(X,\Omega^1)^*$
2) compute $h_p \circ A$ $\rightsquigarrow$ using filtered ϕ-module structure of $D_{cris}(A(x))$.

Lemma. There exists a connection $\mathcal{A}_Z$ with Hodge filtration and Frobenius structure s.t.

$$x^* \mathcal{A}_Z \simeq D_{cris}(A(x))$$

(This follows from Olsson's comparison theorem.)

$\mathcal{A}_Z$ is a unipotent isocrystal, quotient of universal 2-step unipotent conn. $\mathcal{A}_2^{dR}$

suffices to compute

1) Hodge filtration
2) Frobenius structure.

1) Hodge : defined by Hodge filtration on graded pieces and its global nature (Hadian ; universal properties)

2) Frobenius : via Frob. on A_2^{rig} and comparison thm of Chiarellotto - Le Stum ; initial action condition $\rightsquigarrow$ gives a p-adic differential equation that we solve using Tuitman's algorithm

(§§ 5.2 - 5.3 in notes for more details)

Examples of Quadratic Chabauty

A problem of Diophantus (Problem 17, Book VI of Arithmetica :

Find three squares which when added give a square and s.t. the first one is the side (square root) of the second and the second is the side (sq. root) of third: i.e.. can one find positive, rational x, y s.t. $y^2 = x^8 + x^4 + x^2$?

Diophantus found $x = \frac{1}{2}$, $y = \frac{9}{16}$. Are there any others?

Remove the singularity at $(0,0)$ $\rightsquigarrow$ want $X(\mathbb{Q})$ for $X: y^2 = x^6 + x^2 + 1$.

$J(\mathbb{Q})$ has rk 2

$J \sim E_1 \times E_2$, rk $NS(J) = 2$.

Wetherell ('97): determined $X(\mathbb{Q})$ via covering collections and classical Chabauty-Coleman

Bianchi ('19) gave a Q.C.-solution to Diophantus' question using p-adic sigma function

$$X(\mathbb{Q}) = \{\infty^{\pm}, (0, \pm 1), (\pm 1/2, \pm 9/8)\}.$$

(B-Dogra ('16): can apply QC to bielliptic genus 2 curves X/K ($K = \mathbb{Q}$ or quad. imag.) with rk $J(K) = 2$ (computational tools: p-adic heights on elliptic curves, rewrite using double Coleman integrals))

2) $X_0(37)(\mathbb{Q}(i))$: Daniels and Lozano-Robledo

$X_0(37)$: $y^2 = -x^6 - 9x^4 - 11x^2 + 37$

over $\mathbb{Q}(i)$: $J_0(37)(\mathbb{Q}(i))$ has rank 2

B-Dogra-Müller:

$$X_0(37)(\mathbb{Q}(i)) = \{(\pm 2, \pm 1), (\pm i, \pm 4), \infty^{\pm}\}$$

used QC + Mordell-Weil sieve

$p = 41, 73, 101$

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3) $X_s(13)$: the split Cartan curve of level 13

Bilu-Parent ('11): determined Serre Uniformity in split Cartan case

Bilu-Parent-Rebolledo ('13): determined $X_s(\ell)(\mathbb{Q})$ for all $\ell \neq 13$

What about $\ell = 13$?

$g = 3$ curve; model was found by Baran (smooth plane quartic)

rk $NS(J) = 3$
rk $J(\mathbb{Q}) = 3$.

B-Dogra-Müller-Tuitman-Vonk :
$\# X_s(13)(\mathbb{Q}) = 7$

$\xRightarrow{\text{Baran}} \# X_{ns}(13)(\mathbb{Q}) = 7$

4) $X_{S_4}(13)$: $g=3$, smooth plane quartic

Banwait-Cremona · Jacobian is isog. to J of $X_s(13)$

$\# X_{S_4}(13)(\mathbb{Q})_{\text{known}} = 4$

??

B-Dogra-Müller-Tuitman-Vonk: $\# X_{S_4}(13)(\mathbb{Q}) = 4$.

(of interest via Mazur's Program B: last exceptional S_4 curve; last modular curve of level 13^n)

5) Two other curves from Mazur's Program B (via D. Zureick-Brown)

$X_H = X(25)/H \qquad \Gamma(25) \subset H \subset GL_2(\mathbb{Z}_5)$

each ~~have~~ has the following properties:

- 2 known rat'l points
- usual QC hypotheses satisfied.

Fit the global height pairing using the Jacobian and Coleman-Gross p-adic hts on $J(\mathbb{Q})$

BDMTV ('20): $\# X_{11}(\mathbb{Q}) = 2$, $X_{15}(\mathbb{Q}) = 2$ $\Big\}$ used QC + MWS.

6) The curves $X_0(N)^+ := X_0(N)/w_N$

nice curve whose non-cusp~~idal~~ pts classify unordered pairs $\{E_1, E_2\}$ of elliptic curves admitting an N-isogeny.

$X_0(N)^+(\mathbb{Q}) = \{$cusps, CM points, exceptional pts$\}$

restrict to N prime.

Galbraith ('96):

$g(X_0(N)^+) = 2 \Leftrightarrow N \in \{67, 73, 103, 107, 167, 191\}$

$g(X_0(N)^+) = 3 \Leftrightarrow N \in \{97, 109, 113, 127, 139, 149, 151, 179, 239\}$

Y. Hasegawa – K. Hashimoto ('96): $X_0(N)^+$ hyperelliptic $\Leftrightarrow g = 2$.

So $g=3$ curves are smooth plane quartics.

All satisfy the QC hypotheses:

$J := J_0(N)^+$ has RM, so $\operatorname{rk} NS(J) \geq g$

Can show $\operatorname{rk} J(\mathbb{Q}) = g$.

$X_0(N)^+$ has good reduction away from N, but$_3$ does not have potential good reduction at N.

Can show that there's a regular semistable model of $X_0(N)^+$ over $\mathbb{Z}_N$ whose special fiber has a unique irred. component

$\overset{\text{Betts-Dogra}}{\Longrightarrow}$ $h_N = 0$

Galbraith: What are the exceptional points on $X_0(N)^+$ for all such curves of genus ≤ 5?

B-Best-Bianchi-Lawrence-Müller-Triantafillou-Vonk:

$X_0(67)^+(\mathbb{Q})$: no exceptional points

$X_0(73)^+(\mathbb{Q})$, $X_0(103)^+(\mathbb{Q})$: 1 exceptional pt (up to hyperell. inv.)

BDMTV: The only prime values of N st $X_0(N)^+$ is genus 2 or 3 with exceptional rat'l pts are $N = 73, 103, 191$.

(So no exceptional points in $g=3$.)

What about $g=4$ ... or $g=5$?

AWS 2020: looking at this here.