Recall:

$$h(D_1, D_2) = \sum_v h_v(D_1, D_2)$$

$$= \underbrace{\int_{D_2} \omega_{D_1}}_{\overset{\shortparallel}{h_p(D_1, D_2)}} + \sum_{v \neq p} h_v(D_1, D_2)$$

Constructed ω_{D_1} (3rd kind diffs)

How do we compute Coleman integrals of diffs of 3rd kind? $\int_S^R \omega$, $\mathrm{Res}(\omega) = (P) - (Q)$.

1) Compute $\Psi(\omega) \in H^1_{dR}(X)$ by computing cup products

$\longrightarrow \Psi(\omega) = \sum \underline{b_i}\, \omega_i$ for $\{\omega_i\}$ basis of $H^1_{dR}(X)$

solve for b_i

by computing $\Psi(\omega) \cup [\omega_j]$

2) Let $\alpha := \phi^* \omega - p\omega$. Use Frob. equivariance to compute $\Psi(\alpha) = \phi^* \Psi(\omega) - p \cdot \Psi(\omega)$

3) Let β be s.t. $\mathrm{Res}(\beta) = (R) - (S)$. Compute $\Psi(\beta)$.

4) Using coleman reciprocity:

$$\int_S^R \omega = \frac{1}{1-p}\Big(\Psi(\alpha) \cup \Psi(\beta) + \sum_{A \in X(\overline{\mathbb{Q}}_p)} \mathrm{Res}_A \left(\alpha \textstyle\int \beta\right) - \int_{\phi(S)}^{S} \omega - \int_R^{\phi(R)} \omega \Big)$$

This lets us compute $h_p(D_1, D_2)$, since

$$h_p(D_1, D_2) = \int_{D_2} \omega_{D_1}.$$

What about the self-pairing of a divisor?

$h_p(D, D)$??

It turns out that if we consider the case of $X/\mathbb{Q}$ a hyperell. curve w/ odd degree model, genus g

$$h_p(D, D) = -2\sum_{i=0}^{g-1} \int_b^z \omega_i \overline{\omega_i}$$

$\omega_i = \frac{x^i dx}{2y}$

$\overline{\omega_i}$ dual under $\cup$

when $D = (z) - (\infty)$

Can use this to study integral pts on hyperelliptic curves

Quadratic Chabauty for integral pts on hyperell. curves (B-Besser-Müller)

let $f \in \mathbb{Z}[x]$, monic, separable, $\deg 2g+1 \geq 3$

let $\mathcal{U} = \mathrm{Spec}(\mathbb{Z}[x,y]/(y^2-f(x)))$, X be normalization of proj. closure of generic fiber of $\mathcal{U}$.

let J be the Jacobian of X. Assume $\mathrm{rk}\, J(\mathbb{Q}) = g$, suppose $\log: J(\mathbb{Q}) \otimes \mathbb{Q}_p \to H^0(X_{\mathbb{Q}_p}, \Omega^1)^*$ is an isomorphism.

let p be a good prime.

Then $\exists\, \alpha_{ij} \in \mathbb{Q}_p$ s.t.

$$\rho(z) = -2\sum_{i=0}^{g-1} \int_b^z \omega_i \bar{\omega}_i - \sum_{i,j<g} \alpha_{ij} \int_\infty^z \omega_i \int_\infty^z \omega_j,$$

$$\omega_i = \frac{x^i dx}{2y}$$

takes values in an explicitly computable finite set $S \subset \mathbb{Q}_p$ for all $z \in \mathcal{U}(\mathbb{Z})$.

Idea: $\overset{\text{global ht}}{h} = h_p + \sum_{\ell \neq p} h_\ell$

$$\Rightarrow \underbrace{h - h_p}_{\text{Coleman integrals}} = \underbrace{\sum_{\ell \neq p} h_\ell}_{\text{on integral pts, finitely many values, can compute them at the start}}$$

$$h = \sum \underline{\underline{\alpha_{ij}}} \int \omega_i \int \omega_j$$

solve for α_{ij}

What goes wrong for rational points?

We don't know how to control $\sum_{\ell \neq p} h_\ell$ on all rational points.

Goal: Extend QC from integral points to rational points

Problem: Need to control local heights away from p.

Solution: Use height that factors through Kim's unipotent Kummer map, can control local heights in this setting.

- For this, need "non-abelian" height (instead of heights via the Jacobian)
- Use heights on Bloch-Kato Selmer grps

Let $X/\mathbb{Q}$ be a nice curve $g>1$, p good prime.

Let $V = H^1_{et}(X_{\bar{\mathbb{Q}}})^*$

By Nekovář ('93), have a bilinear symmetric pairing

$$h : H^1_f(G_{\mathbb{Q}}, V) \times H^1_f(G_{\mathbb{Q}}, V^*(1)) \to \mathbb{Q}_p$$

where $h = \sum_v h_v$

This is equivalent to the Coleman-Gross height via an étale Abel-Jacobi map (Besser)
This height h also depends on choices, like the C-G height:

1) the choice of an idèle class char
$$\chi : \mathbb{A}_{\mathbb{Q}}^{*}/\mathbb{Q}^{*} \to \mathbb{Q}_p$$

2) a splitting s of the Hodge filtration on $V_{dR} = D_{cris}(V) = H^1_{dR}(X_{\mathbb{Q}_p})^*$

Recall that in the Coleman-Gross height, to pair points on the Jacobian, needed choice of divisors. Here choice depends on <u>mixed extensions</u>:

Given a pair of extn classes
$(e_1, e_2) \in H^1_f(G_{\mathbb{Q}}, V) \times H^1_f(G_{\mathbb{Q}}, V^*(1))$,
take reps E_1, E_2:

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
\mathbb{Q}_p(1) \arrow[r] & E_2 \arrow[r] & V \arrow[r] \arrow[d] & 0 \\
 & & E_1 \arrow[d] & \\
 & & \mathbb{Q}_p \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

fill in $\longrightarrow$ diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathbb{Q}_p(1) \arrow[r] \arrow[d, "\shortparallel"] & E_2 \arrow[r] \arrow[d] & V \arrow[d] \\
0 \arrow[r] & \mathbb{Q}_p(1) \arrow[r] & \boxed{E} \arrow[r] \arrow[d] & E_1 \arrow[d] \\
 & & \mathbb{Q}_p \arrow[r, "="] \arrow[d] & \mathbb{Q}_p \arrow[d] \\
 & & 0 & 0
\end{tikzcd}$$

(G_K-reps)

E is a mixed extn of E_1, E_2 with graded pieces $\mathbb{Q}_p$, V, $\mathbb{Q}_p(1)$.

with a weight filtration

$$0 = W_{-3}E \subseteq W_{-2}E \subseteq W_{-1}E \subseteq W_0E = E \text{ s.t.}$$

$$W_{-1}E \simeq E_2 \qquad W_0E/W_{-2}E = E_{-1}$$

Let $M_{\mathbb{Q}} = \{$ such mixed extensions $\}$

v prime $\rightsquigarrow M_v = \{$ mixed extns of G_v-reps $\}$

$M_{\mathbb{Q},f}$: f subscript : crystalline

For $E \in M_{\mathbb{Q},f}$, define $h_v(E) = h_v(\underset{\in M_v}{\overbrace{\mathrm{loc}_v E}^{E_v \,=}})$

(see lecture notes §4)

and then define

$$h(e_1, e_2) = \sum_v h_v(E)$$

From this point forward, we'll assume that $h_\ell(E_\ell) = 0 \quad \forall \ell \neq p$

(e.g. when X has potential good reduction at ℓ, local height $h_\ell = 0$).

Def. A filtered ϕ-module (over $\mathbb{Q}_p$) is a finite-dim'l $\mathbb{Q}_p$-vector space W, with an exhaustive and separated decreasing filtration Fil^i and an automorphism ϕ:

- exhaustive: $W = \bigcup_i \mathrm{Fil}^i$
- separated: $\bigcap_i \mathrm{Fil}^i = 0$
- decreasing: $\mathrm{Fil}^{i+1} \subseteq \mathrm{Fil}^i$

Examples:

1) $\mathbb{Q}_p$ with $\mathrm{Fil}^0 = \mathbb{Q}_p$, $\mathrm{Fil}^n = 0$ for all $n > 0$, $\phi = \mathrm{id}$

2) By Faltings' comparison theorem, have $H^1_{dR}(X_{\mathbb{Q}_p}) = D_{cris}(H^1_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p))$ and $H^1_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)$ is crystalline, take Frobenius ϕ on crystalline cohomology and Hodge filtration $\Rightarrow H^1_{dR}(X_{\mathbb{Q}_p})$ has the structure of a filtered ϕ-module.

3) $V_{dR} = H^1_{dR}(X_{\mathbb{Q}_p})^* = D_{cris}(V)$ w/ dual filtration and action

4) The direct sum $\mathbb{Q}_p \oplus V \oplus \mathbb{Q}_p(1)$ has the structure of a filtered ϕ-module as well.

Let $E_p \in M_{p,f}$

Then $E_{dR} = D_{cris}(E_p)$ is a mixed extn of filtered ϕ-modules w/ graded pieces $\mathbb{Q}_p, V, \mathbb{Q}_p(1)$.

To construct the local height h_p of E_p, need an explicit description of

- Frobenius ϕ
- filtration on E_{dR}

Want to compute
$$h(E) = \sum h_v(E_v)$$
$$= h_p(E_p) + \cancel{\sum_{\ell \neq p} h_\ell(E_\ell)}$$

From Kim to Nekovář:

Idea: Want maps:

$$X(\mathbb{Q}) \to M_{\mathbb{Q},f}$$
$$X(\mathbb{Q}_p) \to M_{p,f}$$
$$X(\mathbb{Q}_\ell) \to M_\ell$$

factor through unipotent Kummer map

Assume in addition to $X/\mathbb{Q}$ ~~with~~ $g \geq 2$,

$\mathrm{rk}\, J(\mathbb{Q}) = g$, $\mathrm{rk}\, NS(J) > 1$

Then $\exists\, Z \in \mathrm{Pic}(X \times X)$ that allows us to construct a nice quotient $U = U_Z$ of U_2

(by Kim: U_n = n-unipotent quotient of $\pi_1^{et}(X_{\bar{\mathbb{Q}}})$)

(§4-5 in notes)

By work of Kim, have local unipotent Kummer maps

$$j_{U,v} : X(\mathbb{Q}_v) \to H^1(G_v, U)$$

We'll assume that $j_{U,\ell}$ is trivial for all $\ell \neq p$

(in general, by Kim–Tamagawa, know that $j_{U,\ell}$ has finite image)

* this assumption is satisfied in the case of X having everywhere pot. good red.

So we have the following diagram:

$$\begin{tikzcd} X(\mathbb{Q}) \arrow[r] \arrow[d, "j_U =: j"] & X(\mathbb{Q}_p) \arrow[d, "j_{U,p} =: j_p"] \\ H^1_f(G_T, U) \arrow[r, "\mathrm{loc}_p"] & H^1_f(G_p, U) \end{tikzcd}$$

$T = \{\text{bad primes}\} \cup \{p\}$
G_T: max'l quotient of $G_{\mathbb{Q}}$ unr outside T.

Lemma.

The set

$$X(\mathbb{Q}_p)_U = j_p^{-1}(\mathrm{loc}_p(H^1_f(G_T, U))) \text{ is}$$

finite.

More generally, this result holds for $r < g + \mathrm{rk}\, NS(J) - 1$.

We have $X(\mathbb{Q}) \subset X(\mathbb{Q}_p)_U$ and the goal is to compute $X(\mathbb{Q}_p)_U$ using p-adic heights
"quadratic Chabauty" for rat'l points.