Yesterday: gave an alg. to compute Coleman integrals between pts in different residue disks on hyperell. curves "analytic continuation along Frobenius" (wrote down action of Frobenius ϕ on differentials w_i, reduce pole orders, get $\phi^* \omega_i = dh_i + \sum M_{ji} \omega_j$, then write lin system to produce $\left(\begin{matrix} \vdots \\ \int_P^Q \omega_i \\ \vdots \end{matrix} \right)_{i=0,\ldots,2g-1}$)

How do we do this for more general curves?

Use Tuitman's algorithm:

let $X/\mathbb{Q}$ a nice curve of genus g with a plane model

$$Q(x,y) = y^{d_x} + Q_{d_x-1} y^{d_x-1} + \cdots + Q_0 = 0$$

s.t. $Q(x,y)$ irred, $Q_i(x) \in \mathbb{Z}[x]$.

let p be a good prime for X.

1) Consider the map $x : X \to \mathbb{P}^1$ and remove the ramification locus $r(x)$ (analogue of removing Weierstrass pts in Kedlaya's alg.)
2) Choose a lift of Frob with $x \mapsto x^p$, compute image of y through Hensel lifting

3) Compute a basis of $H^1_{dR}(X)$ using integral bases of $\mathbb{Q}(X)$ over $\mathbb{Q}[x]$, $\mathbb{Q}[\frac{1}{x}]$

4) Compute action of Frob. on diffs and reduce pole orders using relations in cohomology (via Lauder's fibration algorithm – Tuitman uses integral bases of $\mathbb{Q}(X)$)

Then $\quad \phi^* \omega_i = dh_i + \sum M_{ji} \omega_j$

Use this to give a lin. system to produce values $\begin{pmatrix} \vdots \\ \int_Q^P \omega_i \\ \vdots \end{pmatrix}_{i=0,\dots,2g-1.}$

Ex (B-Tuitman) Can compute Coleman integrals on a non-hyperell. genus 55 curve to show its Jacobian has rank ≥ 1.

Let $X/\mathbb{Q}$ be a nice curve of genus g.

By work of Coleman ('82) and Coleman-deShalit ('88), have a theory of <u>iterated</u> p-adic integrals on X. These are iterated path integrals:

$$(*) \underbrace{\int_P^Q \eta_n \cdots \eta_1}_{} := \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_n) \cdots f_1(t_1) dt_n \cdots dt_1$$

In our computations, we'll focus on the case $n=2$ (double Coleman integrals):

$$\int_P^Q \eta_2 \eta_1 := \int_P^Q \eta_2(R) \int_P^R \eta_1$$

These integrals play an important role in nonabelian Chabauty.

How do we compute them?

- iteratively apply an algorithm for computing action of Frobenius on p-adic cohomology (e.g. Kedlaya or Tuitman) to produce
$$\phi^* w_i = dh_i + \sum M_{ji} w_j$$
- observe that the eigenvalues of $M^{\otimes n}$ are not 1, and reduce the computation of n-fold iterated integrals to $(n-1)$-fold iterated integrals

Some useful properties of iterated Coleman integrals:

Prop. Let $w_{i_1}, \ldots w_{i_n}$ be forms of the second kind, holomorphic at $P, Q \in X(\mathbb{Q}_p)$

1) $\int_P^P w_{i_1} \cdots w_{i_n} = 0$.

2) $$\sum_{\substack{\text{all} \\ \text{perm } \sigma}} \int_P^Q \omega_{\sigma(i_1)} \cdots \omega_{\sigma(i_n)} = \prod_{j=1}^{n} \int_P^Q \omega_j$$

3) $$\int_P^Q \omega_{i_1} \cdots \omega_{i_n} = (-1)^n \int_Q^P \omega_{i_n} \cdots \omega_{i_1}$$

4) If $P, P', Q \in X(\mathbb{Q}_p)$, then

$$\int_P^Q \omega_{i_1} \cdots \omega_{i_n} = \sum_{j=0}^{n} \int_{P'}^{Q} \omega_{i_1} \cdots \omega_{i_j} \int_P^{P'} \omega_{i_{j+1}} \cdots \omega_{i_n}$$

(This lets us break up a path.)

So this gives the analogue of additivity in endpts for double integrals (have $P, P', Q', Q \in X(\mathbb{Q}_p)$)

$$\int_P^Q \omega_i \omega_k = \int_P^{P'} \omega_i \omega_k + \int_{P'}^{Q'} \omega_i \omega_k + \int_{Q'}^{Q} \omega_i \omega_k$$
$$+ \int_P^{P'} \omega_k \int_{P'}^{Q} \omega_i + \int_{P'}^{Q'} \omega_k \int_{Q'}^{Q} \omega_i$$

let $P' = \phi(P)$, $Q' = \phi(Q)$; here's how we compute double Coleman integrals:

$$\int_{\phi(P)}^{\phi(Q)} \omega_i \omega_k = \int_P^Q \phi^*(\omega_i) \phi^*(\omega_k)$$
$$= \int_P^Q (df_i + \sum M_{ji} \omega_j)(df_k + \sum M_{jk} \omega_j)$$

(involving single integrals) $$= c_{ik} + \int_P^Q \sum M_{ji} \omega_j \sum M_{jk} \omega_j$$

This gives us

$$\begin{pmatrix} \vdots \\ \int_P^Q \omega_i\omega_k \\ \vdots \end{pmatrix} = \left(I - M^{t\otimes 2}\right)^{-1} \begin{pmatrix} C_{ik} - \int_{\phi(P)}^{P} \omega_i\omega_k \\ - \int_P^Q \omega_i \int_{\phi(P)}^{P} \omega_k \\ - \int_{\phi(P)}^{\phi(Q)} \omega_i \int_{\phi(P)}^{\phi(Q)} \omega_k \\ + \int_{\phi(Q)}^{Q} \omega_i\omega_k \end{pmatrix}$$

Application (preview) Let $\mathcal{E}/\mathbb{Z}$ be the minimal regular model of an elliptic curve. Let $\mathcal{X} = \mathcal{E} \setminus O$. Let $\omega_0 = \dfrac{dx}{2y + a_1x + a_3}$, $\omega_1 = x\omega_0$. Let b be a tangential basept at O or an integral 2-torsion pt. Let p be a prime of good reduction. Suppose $\mathcal{E}$ has analytic rk 1 and Tamagawa product 1. Let $\log(z) = \int_b^z \omega_0$, $D_2(z) = \int_b^z \omega_0\omega_1$

Thm (Kim, B-Kedlaya-Kim) Suppose P is a pt. of infinite order in $\mathcal{E}(\mathbb{Z})$. Then $\mathcal{X}(\mathbb{Z}) \subseteq \mathcal{E}(\mathbb{Z})$ is in the zero set of

$$f(z) = (\log(P))^2 D_2(z) - (\log(z))^2 D_2(P).$$

Kim: this D_2 is related to a p-adic height on $\mathcal{E}$!

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p-adic heights on Jacobians of curves

p-adic heights are a natural source of bilinear forms on global pts, allow us to generalize some of our linear techniques from Chabauty–Coleman

Rmk. same p-adic heights as in p-adic BSD / p-adic GZ

Let $X/\mathbb{Q}$ be a nice curve of genus $g \geq 1$, p a good prime.

Fix a branch of $\log_p : \mathbb{Q}_p^* \to \mathbb{Q}_p$. Also fix:

1) an idèle class char. $\chi : \mathbb{A}_{\mathbb{Q}}^* / \mathbb{Q}^* \to \mathbb{Q}_p$
2) a splitting s of the Hodge fil. on $H^1_{dR}(X/\mathbb{Q}_p)$ such that the $\ker(s)$ is isotropic wrt the cup product

Fixing A splitting of the Hodge fil. corresponds to fixing a subspace $W = \ker(s)$ of $H^1_{dR}(X)$ complementary to the space $H^0(X, \Omega^1)$, i.e., $H^1_{dR}(X/\mathbb{Q}_p) \simeq H^0(X, \Omega^1) \oplus W$

Def (Coleman-Gross '89) The cyclotomic p-adic height pairing is a symmetric bi-additive pairing

$$\mathrm{Div}^0(X) \times \mathrm{Div}^0(X) \to \mathbb{Q}_p$$
$$(D_1, D_2) \mapsto h(D_1, D_2) \text{ for}$$

$D_1, D_2 \in \mathrm{Div}^0(X)$ with disjoint support

s.t.

1) $h(D_1, D_2) = \sum_{\text{finite } v} h_v(D_1, D_2)$

$$= h_p(D_1, D_2) + \sum_{\ell \neq p} h_\ell(D_1, D_2)$$

$$= \int_{D_2} \omega_{D_1} + \sum m_\ell \log_p(\ell)$$

$m_\ell \in \mathbb{Q}$ is an intersection mult.

2) For $\beta \in \mathbb{Q}(X)^*$, have $h(D, \mathrm{div}(\beta)) = 0$, so gives a symmetric bilinear pairing

$$J(\mathbb{Q}) \times J(\mathbb{Q}) \to \mathbb{Q}_p .$$

Local height at p.

Need to construct a normalized differential ω_{D_1} wrt choice of W

let $T(\mathbb{Q}_p)$ be diffs of 3rd kind : simple poles and <u>integer</u> residues.

Have residue divisor hom:

$$\mathrm{Res}: T(\mathbb{Q}_p) \to \mathrm{Div}^0(X)$$
$$\omega \longmapsto \mathrm{Res}(\omega) = \sum_P (\mathrm{Res}_P\,\omega) P,$$

Induces

$$0 \to H^0(X_{\mathbb{Q}_p}, \Omega^1) \to T(\mathbb{Q}_p) \xrightarrow{\mathrm{Res}} \mathrm{Div}^0(X) \to 0.$$

Want : ω_{D_1} will be a certain 3rd kind diff with $\mathrm{Res}(\omega_{D_1}) = D_1$.

Ex. X hyperell. curve $y^2 = f(x)$, $D_1 = (P) - (Q)$, P, Q non-Weier pts.

Then $\omega = \frac{dx}{2y}\left(\frac{y + y(P)}{x - x(P)} - \frac{y + y(Q)}{x - x(Q)}\right)$ has

Res div. D_1 : simple poles at P, Q, residues $+1$, -1, resp; However adding any holomorphic η to ω and taking $\mathrm{Res}(\eta + \omega) = D_1$. So must take care of this!

Let $T_\ell(\mathbb{Q}_p)$ be log diffs: $\frac{df}{f}$, $f \in \mathbb{Q}_p(X)^*$

We have

$$0 \to H^0(X_{\mathbb{Q}_p}, \Omega^1) \to T(\mathbb{Q}_p)/T_\ell(\mathbb{Q}_p) \to J(\mathbb{Q}_p) \to 0$$

Prop. There is a canonical hom. $\Psi : T(\mathbb{Q}_p)/T_\ell(\mathbb{Q}_p) \to H^1_{dR}(X)$

s.t. 1) Ψ is the identity on hol. diffs

2) Ψ sends third kind diffs to second kind mod exact diffs.

Def Let $D \in \mathrm{Div}^0(X)$. Then ω_D is the unique diff. of the third kind with $\mathrm{Res}(\omega_D) = D$ and $\Psi(\omega_D) \in W$.

Rmk. If p is ordinary, we can take W to be the unit root subspace for Frobenius.