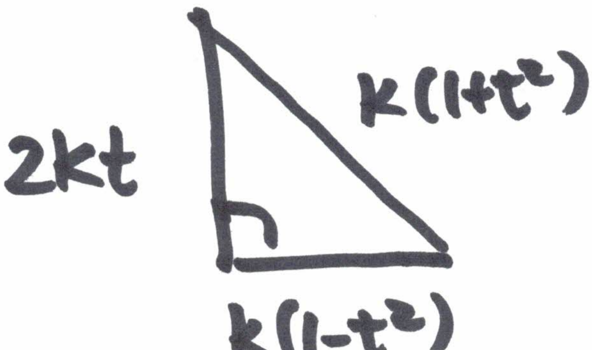
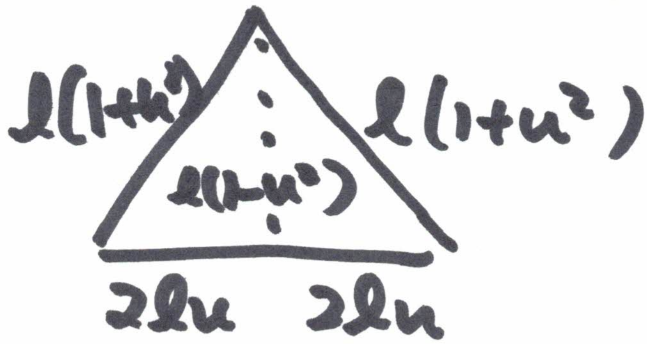
Question: Does there exist a pair of a rat'l right triangle and a rat'l isosceles triangle that have the same area and the same perimeter?





Rescale: $l=1$; suppose $k, t, u \in \mathbb{Q}$

$0<t, u<1$, $k>0$

Equate areas and perimeters:

$$\begin{cases} k^2 t(1-t^2) = 2u(1-u^2) \\ k+kt = 1+2u+u^2 \end{cases}$$

some algebra $\Rightarrow$ $\exists x \in \mathbb{Q}$, $1<x<2$ s.t.

$$2xk^2 + (-3x^3-2x^2+6x-4)k + x^5 = 0$$

Discriminant of poly. in k must be rat'l square:

$$X: \quad y^2 = (-3x^3 - 2x^2 + 6x - 4)^2 - 4(2x)x^5$$
$$= x^6 + 12x^5 - 32x^4 + 52x^2 - 48x + 16$$

This is a genus 2 curve, and we'd like to determine $X(\mathbb{Q})$.

The Jacobian J of X has $\operatorname{rk} J(\mathbb{Q}) = 1$

Also, Chabauty-Coleman bound gives

$$\# X(\mathbb{Q}) \leq 10$$

We can find

$$\{\infty^{\pm}, (0, \pm 4), (1, \pm 1), (2, \pm 8),$$
$$(12/11, \pm 868/11^3)\} \subseteq X(\mathbb{Q})$$

We've found 10 rat'l points!

Answer to Δ question:

Thm (Hirakawa-Matsumura '18) Yes, exactly one pair of triangles!

Coleman's effective Chabauty
("Chabauty-Coleman bound")

Let $X/\mathbb{Q}$ be a nice curve, with $g \geq 2$.
Suppose $\mathrm{rk}\, J(\mathbb{Q}) < g$. If $p > 2g$ is good:
$$\# X(\mathbb{Q}) \leq \# X(\mathbb{F}_p) + 2g - 2.$$

This bound comes from bounding the number of zeros of a p-adic (Coleman) integral.

Coleman: gave theory of p-adic line integration in 1980s.

Thm (Coleman) Let $X/\mathbb{Q}_p$ be a nice curve with good reduction at p.
The p-adic integral $\int_P^Q \omega \in \overline{\mathbb{Q}}_p$, defined for $P, Q \in X(\overline{\mathbb{Q}}_p)$ and $\omega \in H^0(X, \Omega^1)$ satisfies the following:

1) the integral is $\overline{\mathbb{Q}}_p$-linear in ω

2) if P, Q reduce to the same point $\bar{P} \in X(\overline{\mathbb{F}}_p)$ then we call the integral a tiny integral.

3) We have

$$\int_P^Q \omega + \int_{P'}^{Q'} \omega = \int_P^{Q'} \omega + \int_{P'}^{Q} \omega$$

$\Rightarrow$ can define $\int_D \omega$ for $D = \sum_{j=1}^{n} ((Q_j)-(P_j)) \in \mathrm{Div}^0_X(\overline{\mathbb{Q}}_p)$

as $$\int_D \omega = \sum_{j=1}^{n} \int_{P_j}^{Q_j} \omega$$

4) D principal $\Rightarrow \int_D \omega = 0$

5) Integral compatible w/ $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-action

6) Fix $P_0 \in X(\overline{\mathbb{Q}}_p)$. If $0 \neq \omega \in H^0(X_{\mathbb{Q}_p}, \Omega^1)$, then the set of pts $P \in X(\overline{\mathbb{Q}}_p)$ reduce to a fixed pt on $X(\overline{\mathbb{F}}_p)$ s.t. $\int_{P_0}^{P} \omega = 0$ is finite.

This is the Coleman integral.

Cor. Given hypotheses of previous thm, let $b \in X(\mathbb{Q}_p)$, $i : X \hookrightarrow J$

$$P \mapsto [P-b]$$

There is a map $J(\mathbb{Q}_p) \times H^0(X_{\mathbb{Q}_p}, \Omega^1) \to \mathbb{Q}_p$

$$(Q, \omega) \mapsto \langle Q, \omega \rangle$$

that's additive in Q, $\mathbb{Q}_p$-linear in ω, and given by $\langle [D], \omega \rangle = \int_D \omega$ for $D \in \mathrm{Div}_X^0$.

For $P \in X(\mathbb{Q}_p)$, we have the Abel-Jacobi morphism AJ_b that takes P to

$$\langle i(P), \omega \rangle = \int_b^P \omega =: AJ_b(P).$$

The Chabauty-Coleman method uses a certain subspace of the space of reg. 1-forms; now assume $b \in X(\mathbb{Q})$, use it to embed $X \hookrightarrow J$.

Def. Let $A = \{\omega \in H^0(X, \Omega^1) : \text{for all } P \in J(\mathbb{Q})$ $\langle P, \omega \rangle = 0\}$ be the subspace of annihilating differentials.

We have:

$$\begin{tikzcd} X(\mathbb{Q}) \arrow[r] \arrow[d] & X(\mathbb{Q}_p) \arrow[d] \arrow[drr, "AJ_b"] & & \\ J(\mathbb{Q}) \arrow[r] & J(\mathbb{Q}_p) \arrow[r, "\log"] & H^0(J_{\mathbb{Q}_p}, \Omega^1)^* \arrow[r, phantom, "\simeq"] & H^0(X_{\mathbb{Q}_p}, \Omega^1)^* \end{tikzcd}$$

By "computing rat'l pts via Chabauty Coleman": compute the finite set of p-adic pts

$$X(\mathbb{Q}_p)_1 := \left\{ z \in X(\mathbb{Q}_p) : \int_b^z \omega = 0 \text{ for } \omega \in A \right\}$$

By construction, $X(\mathbb{Q}) \subset X(\mathbb{Q}_p)_1$.

How do we compute annihilating diffs?

Ex. let $X: y^2 = x^5 - 2x^3 + x + \frac{1}{4}$

(LMFDB: 971.a.971.1)

Some facts about X:

1) $X(\mathbb{Q})_{\text{known}} = \{ \infty, (0, \pm 1/2), (-1, \pm 1/2), (1, \pm 1/2) \}$

2) J is simple, $J(\mathbb{Q}) \simeq \mathbb{Z}$.

$[(-1,-\frac{1}{2}) - (0,\frac{1}{2})] \in J(\mathbb{Q})$
has infinite order.

3) X is good at $p=3$, $\#X(\mathbb{F}_3) = 7$

Stoll's refinement of Chabauty-Coleman for $p=3$:

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2\cdot r + \left\lfloor \frac{2r}{p-2} \right\rfloor = 11.$$

So need to do more work to determine $X(\mathbb{Q})$ here.

We'll construct a 3-adic annihilating differential η.

Basis of $H^0(X_{\mathbb{Q}_3}, \Omega^1)$ is $\left\{ \omega_i = \frac{x^i\,dx}{2y} \right\}_{i=0,1}$

So η is a $\mathbb{Q}_3$-lin. comb. of ω_0, ω_1

We'll compute the values of

$\alpha := \int_{(0,1/2)}^{(-1,-1/2)} \omega_0$ and $\beta := \int_{(0,1/2)}^{(-1,-1/2)} \omega_1$ to compute η.

SageMath can compute α, β:

$$\alpha = 3 + 3^2 + 3^4 + \cdots$$

$$\beta = 2 + 2\cdot 3 + 2\cdot 3^3 + \cdots$$

We take $\eta = \beta\omega_0 - \alpha\omega_1$ and run Chabauty–Coleman.

Where do these numbers come from?

<u>Explicit Coleman integration</u> using the action of Frobenius on p-adic cohomology

(Sage for hyperell. curves/ Magma for plane curves)

Let X^{an} denote the rigid analytic space over $\mathbb{Q}_p$ associated to $X/\mathbb{Q}_p$.

A wide open subspace of X^{an}: the complement in X^{an} of the union of a finite collection of disjoint closed disks of radius < 1.

<u>More properties of Coleman integral:</u>

Thm (Coleman) Let η, ξ be 1-forms on a wide open V of X^{an}, $P, Q, R \in V(\overline{\mathbb{Q}}_p)$, let $a, b \in \overline{\mathbb{Q}}_p$. Then we have

1) linearity in integrand:

$$\int_P^Q a\eta + b\xi = a\int_P^Q \eta + b\int_P^Q \xi$$

2) additivity in end pts:

$$\int_P^R \eta + \int_R^Q \eta = \int_P^Q \eta$$

3) change of variables under rigid analytic maps (Frobenius)

4) fundamental theorem of calculus

$$\int_Q^P df = f(P) - f(Q)$$

5) Galois compatibility.

We first integrate $\int_P^Q \omega$ for ω 1-form of 2nd kind, $P, Q \in V(\mathbb{Q}_p)$

Suppose X is a hyperell. curve

Sketch of explicit Coleman integration
(B - Bradshaw - Kedlaya)

1) take ϕ a lift of p-power Frobenius
2) compute a basis $\{\omega_i\}$ of 1-forms of 2nd kind
3) Compute $\phi^* \omega_i$ via Kedlaya's zeta function algorithm and use properties of Coleman integral to relate
$\int_P^Q \phi^* \omega_i$ to $\int_P^Q \omega_i$, as well as other easier terms.
4) Solve for $\int_P^Q \omega_i$ using lin. alg.

Kedlaya's algorithm (sketch)

Let $X: \; y^2 = P(x)$

- work in an affine $Y \subset X$, given by deleting Weierstrass pts.

- take ϕ to be:

$$x \longmapsto x^p$$

$$y \longmapsto y^p \sum_{j=0}^{\infty} \binom{1/2}{j} \left(\frac{P(x^p) - P(x)^p}{y^{2p}} \right)^j$$

- Compute the action of ϕ on

$$\phi^*\left(\frac{x^i dx}{y}\right) = \frac{x^{pi} d(x^p)}{\phi(y)} = \frac{x^{pi} p x^{p-1} dx}{\phi(y)}$$

$$= p x^{pi+p-1} \, y^{-p} \cdot \sum_{j=0}^{\infty} \binom{-1/2}{j} \left(\underline{\qquad\qquad} \right)^j$$

and reduce pole order of each resulting differential using relations in H^1.

Denote the basis by $\{w_i\}_{i=0,\dots,2g-1}$:

Kedlaya's algorithm gives

$$\phi^* w_i = dh_i + \sum_{j=0}^{2g-1} M_{ji} w_j$$

If we can compute h_i and M, then:

$$\begin{pmatrix} \vdots \\ \int_Q^P w_i \\ \vdots \end{pmatrix} = (M^t - I)^{-1} \begin{pmatrix} \vdots \\ h_i(P) - h_i(Q) - \int_P^{\phi(P)} w_i - \int_{\phi(Q)}^{Q} w_i \\ \vdots \end{pmatrix}$$

Finishing w/ the 3-adic integrals on

$y^2 = x^5 - 2x^3 + x + \frac{1}{4}$:

- We constructed $\eta = \beta w_0 - \alpha w_1$, where α, β are computed using $(*)$.
- We want to compute $X(\mathbb{Q}_3)$,

Compute power series expansions of $\left\{ \int_{(0,1/2)}^{P_t} \eta \right\}$ where P_t ranges over all residue disks :

$$\int_{(0,1/2)}^{P_t} \eta = \underbrace{\int_{(0,1/2)}^{P_0} \eta}_{\text{3-adic \#}} + \underbrace{\int_{P_0}^{P_t} \eta}_{\text{3-adic series.}}$$

Lucky fact : for each residue disk, $\exists\ P_0 \in X(\mathbb{Q})$; the 3-adic # is 0.

Computing the tiny integral in each residue disk, we find each just has a simple zero at known rat'l point.

This proves that $\# X(\mathbb{Q}) = 7$.