Euler class

Algebraic topology: X $\mathbb{R}$-mfld dim d

$V \longrightarrow X$ be a rank r vector bundle

<u>Def</u>: V is <u>oriented</u> by u a Thom class i.e. $u \in H^r(\cancel{X},\, Th(V), \mathbb{Z})$ which when restricted to $H^r(Th(V_x), \mathbb{Z})$ is a generator

<u>Recall</u>: $Th(V) \cong \mathbb{P}(V \oplus \theta) / \mathbb{P}(V) \simeq V / V - X$

$$Th(V_x) \simeq \mathbb{P}(V_x \oplus \theta) / \mathbb{P}(V_x) \simeq S^r$$

<u>Ex</u>: $\mathcal{U}$ open cover X.

V is described by clutching functions

$\{ \varphi_{U \cap W} : U, W \in \mathcal{U} \}$ s.t. $\det \varphi_{U \cap W} > 0$

$\iff \det V \cong L^{\otimes 2}$ $\quad L \to X$ is a line bundle

<u>Def</u>: X is oriented if TX is

Assume X or, mfld compact $d = r$, $e(V) \in \mathbb{Z}$

Poincaré duality $H^d(X, \mathbb{Z}) \cong \mathbb{Z}$

Compute $e(V)$:

Choose section σ with only isolated zeros

$$e(V) = \sum_{\substack{x \in X \\ \sigma(x)=0}} \deg_x \sigma$$

where σ is locally identified with a function

$$\mathbb{R}^d \longrightarrow \mathbb{R}^r$$

by choosing local coords and local trivializations compatible with orientations

Rmk: If change <u>both</u> local coords and local triv by matrix with $\det < 0$, $\deg_x \sigma$ does not change

Def: $V \to X$ is <u>relatively oriented</u> if

$$\mathrm{Hom}(\det TX, \det V)$$

is oriented.

Def: Let $\mathcal{O}(V)$ be local system on X with $\mathcal{O}(V)_x = H^r(Th(V_x), \mathbb{Z})$

Have $e(V) \in H^r(X, \mathcal{O}(V))$

when $V \to X$ is relatively oriented, we again have $e(V) \in \mathbb{Z}$

$\mathbb{A}^1$-alg top: X in Sm_k of dim d

$V \to X$ alg bundle rank r

Def: V is oriented by the data of $L \to X$ line bundle and iso

$$\det V \cong L^{\otimes 2}$$

Def: V is relatively oriented as before

Ex: $X = \mathbb{P}^n$, $Gr(m,n)$ = parametrizing $(\mathbb{P}^m)$'s in $\mathbb{P}^n$

$\det TX = \mathcal{O}(n+1)$

X is orientable $\Leftrightarrow$ n is odd

<u>Ex</u>: $\mathcal{O}(n) \to \mathbb{P}^1$ is relatively orientable $\Leftrightarrow$ n even

<u>Ex</u>: $\mathcal{O}(d) \oplus \mathcal{O}(e) \to \mathbb{P}^2$ is relatively orientable $\Leftrightarrow$ $d+e$ is ~~even~~ odd

$\rightsquigarrow$ Enriched Bēzout's theorem
(S. McKean)

<u>Euler class</u>: X sm, proper dim $d=r$
(perspective joint with Jesse Kass)
$V \to X$ rank r

σ section of V with only isolated zeros

$$e(V) = \sum_{\substack{x \in X \\ \sigma(x)=0}} \deg_x \sigma \in GW(k)$$

to define: $\deg_x \sigma$

Def: Nisnevich coords near x are

$$\begin{array}{c} \varphi: U \longrightarrow \mathbb{A}^d \quad \text{étale} \\ \downarrow \\ p \end{array}$$

s.t. $\kappa(\varphi(p)) \overset{\cong}{\subseteq} \kappa(p)$

- Such coords determine a distinguished section of $\det TX(U)$
- A local trivialization $\psi: V|_U \to \mathcal{O}_U^r$ determines a distinguished section of
$\det V(U)$

Def: local coords and local trivialization are compatible if distinguished section of $\operatorname{Hom}(\det TX, \det V) \cong L^{\otimes 2}$ is a tensor square.

Spose φ and ψ are compatible

Assumption: if $\varphi : U \subset \mathbb{A}^d$, then σ can be identified with

$$\mathbb{A}^d \xrightarrow{\sigma} \mathbb{A}^r$$

$$\deg_p \sigma := \deg_{\varphi(p)} \sigma$$

<u>Rmk</u>: <u>Assumption</u> is not necessary by finite determinacy of $\deg_p$

- well-defined under conditions

Barge-Morel: $e(V) \in \widetilde{CH}^r(X, \det(-V))$

$\langle 1 \rangle \in \widetilde{CH}^0(X)$

$$\to \bigoplus_{z \in X^{(0)}} GW(k(z), \det T_z X) \to \cancel{\bigoplus_{z \in X^{(1)}}}$$

$$\begin{tikzcd} V \arrow[r, "p"] & X \arrow[l, bend left, "\sigma"] \end{tikzcd}$$ zero section

$$\sigma_* : \widetilde{CH}^0(X) \longrightarrow \widetilde{CH}^r(V, \det p^*V)$$

$$p^* : \widetilde{CH}^r(X, \det V) \to \widetilde{CH}^r(V, \det p^*V)$$

$$e(V) = (p^*)^{-1}\sigma_* \langle 1 \rangle$$

when $\begin{tikzcd} V \arrow[r] & X \arrow[d, "\pi"] \\ & \operatorname{Spec} k \end{tikzcd}$ is relatively oriented

$$\pi_* e(V) \in GW(k)$$

<u>Ex</u>: n even $\quad e(\mathcal{O}_{\mathbb{P}^1}(n)) = \deg_0 x^n$

$$= \frac{n}{2}(\langle 1 \rangle + \langle -1 \rangle)$$

Q: How many lines meet 4 general lines in $\mathbb{P}^3$?

joint with P. Srinivasan, c.f. Matthias Wendt

$Gr(1,3)$ parametrizes lines in $\mathbb{P}3$

equivalently

$W \subseteq k^{\oplus 4}$ $\dim W = 2$

L_1, L_2, L_3, L_4 be 4 lines no two of which intersect

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of k^4 s.t. $L_1 = \mathbb{P}(ke_3 \oplus ke_4)$
$= \{\phi_1 = \phi_2 = 0\}$

Let $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ be dual basis

$L = \mathbb{P}(k\tilde{e}_3 \oplus k\tilde{e}_4)$ $\tilde{e}_3, \tilde{e}_4 \in k^4$ linearly independent

$L \cap L_1 \neq \emptyset \iff (\phi_1 \wedge \phi_2)(\tilde{e}_3 \wedge \tilde{e}_4)$
$= 0$

$S^* \wedge S^* \longrightarrow Gr(1,3)$
be line bundle

$S^* \wedge S^*_{\mathbb{P}W} = W^* \wedge W^*$

Then L_1 determines a section σ_1 of $S^* \wedge S^*$
$\{(\phi_1, \phi_2)$

by $\sigma_1(\mathbb{P}W) = \phi_1|_W \wedge \phi_2|_W$

$\{$ lines intersecting $L_1 \} = \{$ zeros of $\sigma_1 \}$

Using L_2, L_3, L_4, form analogous section

σ of $\bigoplus_{i=1}^{4} S^* \wedge S^* =: V$

Then $\{ \sigma = 0 \} = \{ L : L \cap L_i \neq \emptyset$
$i = 1, \ldots, 4 \}$

<u>Q</u>: Is V relatively orientable?

$\det TX = \mathcal{O}(4)$

$\det V = (S^* \wedge S^*)^{\otimes 4}$

<u>A</u>: yes

computing $\deg_{\mathbb{P}W} \sigma$:

. choose coords on $Gr(1,3)$

$\tilde{e}_1 = e_1$

$\tilde{e}_2 = e_2$

$\tilde{e}_3 = x e_1 + y e_2 + e_3$

$\tilde{e}_4 = x' e_1 + y' e_2 + e_4$

Let $\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3, \tilde{\phi}_4$ be dual basis

$Gr(1,3) \supset U = \operatorname{Spec} k[x, y, x', y']$

$\mathbb{P}(k\tilde{e}_3 \oplus k\tilde{e}_4) \longleftrightarrow (x, y, x', y')$

• $S^* \wedge S^*$ is locally trivialized by $\tilde{\phi}_3 \wedge \tilde{\phi}_4$

write σ as a function $\mathbb{A}^4 \to \mathbb{A}^4$

$$L_1 = \mathbb{P}(k e_3 \oplus k e_4)$$

$$\sigma = (\underset{\phi_1 \wedge \phi_2}{\underset{\|}{\sigma_1}}, ?, ?, ?)$$

didn't bother making notation

$$\phi_1 \wedge \phi_2 \Big|_{k\tilde{e}_3 \oplus \tilde{e}_4} = (x\tilde{\phi}_3 + y\tilde{\phi}_4) \wedge (x'\tilde{\phi}_3 + y'\tilde{\phi}_4)$$

$$= (xy' - yx')\,\tilde{\phi}_3 \wedge \tilde{\phi}_4$$

$$\sigma(x, y, x', y') = (xy' - yx', ?, ?, ?)$$

compute local degree...

<u>Q</u>: Is there an arithmetic-geometric interpretation of $\deg_{P=L} \sigma$?

<u>Q</u>: What arithmetic-geometric information is available?

$L = \mathbb{P}W$ is a line intersecting L_1, L_2, L_3, L_4

$\{ L \cap L_i : i = 1, \dots, 4 \}$ is 4 pts on $L \cong \mathbb{P}^1_{k(L)}$

Let $\lambda =$ cross-ratio

Planes in $\mathbb{P}^3$ containing L are $\mathbb{P}^1_{k(L)}$

dim 3 subspaces V containing W

$$W \subseteq V \subseteq k(L)^4$$

$$\underset{\uparrow \text{dim } 1}{\overline{V}} \subseteq k(L)^2$$

$\{ \operatorname{span}(L, L_i) : i = 1,2,3,4 \}$ is 4 points on $\mathbb{P}^1_{k(L)}$

Let μ = cross-ratio

$$\deg_L \sigma = \mathrm{Tr}_{k(L)/k} \langle \lambda - \mu \rangle$$

<u>Thm</u> (Srinivasan, W.) Let $\overset{\text{char} \neq 2}{}$ L_1, L_2, L_3, L_4 be pairwise nonintersecting lines in $\mathbb{P}^3$

$$\sum_{\substack{L \text{ s.t.} \\ L \cap L_i \neq \emptyset}} \mathrm{Tr}_{k(L)/k} \langle L - \mu \rangle = \langle 1 \rangle + \langle -1 \rangle$$