User's guide to $\mathbb{A}^1$-homotopy theory

want: $\mathbb{P}^n/\mathbb{P}^{n-1}$, colimit

<u>ex.</u>:

$$\begin{tikzcd} \mathbb{P}^{n-1} \arrow[r] \arrow[d] & \mathbb{P}^n \arrow[d] \\ * \arrow[r] & \mathbb{P}^n/\mathbb{P}^{n-1} \end{tikzcd}$$

<u>ex 2</u>: open sets U, V

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\ V \arrow[r] & U \cup V \end{tikzcd}$$

want to glue, crush schemes like topological spaces

treat smooth schemes like manifolds

construction of $\mathbb{A}^1$-homotopy theory (Morel-Voevodsky)

Sm_k = smooth schemes / k

$$Sm_k \longrightarrow \mathrm{Func}(Sm_k^{op}, sset)$$

topological spaces

$$Y \longmapsto \mathrm{Mor}(-, Y)$$

~~hmm~~

homotopy theory can mean:

simplicial model category

or

∞-category

$\mathrm{Pre}(Sm_k) = \mathrm{Func}(Sm_k^{op}, sSet)$

freely adding colimits

Problem: had colimits from 2 in Sm_k

Fix: Force certain classes of maps to be weak equivalences

Bousfield localization

For an open cover $U = \coprod_\alpha U_\alpha \to X$

force $\mathrm{cosk}^0_X \coprod_\alpha U_\alpha \xrightarrow{\sim} X$

$$\mathrm{Pre}(Sm_k) \xrightarrow{L_\tau} Sh_k$$

τ Grothendieck topology

Choices:

Zariski, Nisnevich, étale

$\xrightarrow{\hspace{3em}}$
more open sets

<u>Def</u>: $f: X \to Y$ map Sm_k
is étale at x if

$$T_x X \xrightarrow{\cong} T_{f(x)} Y$$

<u>Def</u>: $U = \coprod_\alpha U_\alpha \to X$ is an étale cover
if it is étale and surjective

<u>Def</u>: $U = \coprod_\alpha U_\alpha \longrightarrow X$ is a Nisnevich
cover if it is an étale cover
and for every $x \in X$ $\exists$ $u \in U$ s.t.
$u \mapsto x$, $k(x) \xrightarrow{\cong} k(u)$

Nice properties:

- $Z \hookrightarrow X$ in Sm_k can often be viewed as $\mathbb{A}^d \to \mathbb{A}^n$

$\vdots$

$$\begin{tikzcd} Sm_k \arrow[r] & PSh_k \arrow[r, "L_{Nis}"] \arrow[d, phantom, "\shortparallel"] & Sh_k \arrow[r, "L_{\mathbb{A}^1}"] & Spc_k \\ & \mathrm{Func}(Sm_k^{op}, sSet) & & \end{tikzcd}$$

force $X \times \mathbb{A}^1 \xrightarrow{\sim} X$

Spc_k is $\mathbb{A}^1$-homotopy thy

<u>Spheres</u>

<u>Def.</u> Given pointed spaces X and Y

$$X \wedge Y := \frac{X \times Y}{X \times * \cup * \times Y}$$

ex: $S^n \wedge S^m = S^{n+m}$

Spheres: S^1, $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$

$$S^{p+q\alpha} = S^{p+q,q} = (S^1)^{\wedge p} \wedge (\mathbb{G}_m)^{\wedge q}$$

Ex:
$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{A}^1 \simeq * \arrow[d] \\
\mathbb{A}^1 \arrow[r] & \mathbb{P}^1
\end{tikzcd}$$
$\mathbb{A}^1 \simeq *$

$$\Rightarrow \quad \mathbb{P}^1 \simeq \Sigma \mathbb{G}_m = S^1 \wedge \mathbb{G}_m$$

Ex: $\mathbb{A}^n - \{0\} \simeq (S^1)^{\wedge n-1} \wedge (\mathbb{G}_m)^{\wedge n}$

induction and
$$\begin{tikzcd}
(\mathbb{A}^{n-1} - \{0\}) \times (\mathbb{A}^1 - \{0\}) \arrow[r] \arrow[d] & (\mathbb{A}^{n-1} - \{0\}) \cancel{\times \mathbb{A}^1} \arrow[d] \\
\cancel{\mathbb{A}^n} \times (\mathbb{A}^1 - 0) \arrow[r] & \mathbb{A}^n - \{0\}
\end{tikzcd}$$

$$\begin{array}{ccc} X \times Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \Sigma X \wedge Y \end{array} \qquad \square$$

<u>Ex:</u> $\mathbb{P}^n / \mathbb{P}^{n-1} \simeq (S^1)^{\wedge n} \wedge (\mathbb{G}_m)^{\wedge n}$

$$\mathbb{P}^n / \mathbb{P}^{n-1} \simeq \frac{\mathbb{P}^n}{\mathbb{P}^n - \{0\}} \simeq \frac{\mathbb{A}^n}{\mathbb{A}^n - \{0\}}$$

$$\simeq * / \mathbb{A}^n - \{0\} \simeq \Sigma(\mathbb{A}^n - \{0\})$$

<u>Thom spaces</u> Let $V \to X$ alg vector bundle

$$Th(V) = V / V - X \simeq \frac{\mathbb{P}(V \oplus \overset{\text{trivial line bundle}}{\mathcal{O}})}{\mathbb{P}(V)}$$

$S \in sSet$

$S \in Pre(Sm_k) = Fun(Sm_k^{op}, sSet)$

<u>Purity thm</u>: $Z \hookrightarrow X$ closed immersion in Sm_k

$$X/_{X-Z} \simeq Th(N_Z X)$$

<u>Ex</u>: $\operatorname{Spec} k \overset{z}{\hookrightarrow} X \circlearrowleft sm$

U open nbhd of z

$$U/_{U-z} \simeq \mathbb{P}^n/\mathbb{P}^{n-1}$$

<u>Ex</u>: $\operatorname{Spec} k(z) \overset{z}{\hookrightarrow} X$

U open nbhd of z

$$U/_{U-z} \simeq \mathbb{P}^n_{k(z)} / \mathbb{P}^{n-1}_{k(z)} \simeq \mathbb{P}^n/\mathbb{P}^{n-1} \wedge (\operatorname{Spec} k(z))_+$$

compare: z pt on a manifold
U small ball around z

$$\Sigma \partial U \simeq U / U - z$$

$GW(k)$, $K_*^M(k)$:

$GW(k) =$ group completion of isomorphism classes of symmetric, nondegenerate bilinear forms over k

$\otimes$ gives ring structure

generators: $\langle a \rangle \quad a \in k^*$

$\langle a \rangle : k \times k \longrightarrow k$

$(x, y) \longmapsto axy$

relations:

- $\langle ab^2 \rangle = \langle a \rangle \quad b \in k^*$
- $\langle a \rangle \langle b \rangle = \langle ab \rangle$
- $\langle a \rangle + \langle b \rangle = \langle a+b \rangle + \langle ab(a+b) \rangle \quad a+b \neq 0$

$$\Rightarrow h := \langle 1 \rangle + \langle -1 \rangle = \langle a \rangle + \langle -a \rangle \quad \forall a$$

hyperbolic form

$$\text{rank} : GW(k) \longrightarrow \mathbb{Z}$$
$$B : V \times V \to k \longmapsto \dim V$$

Fundamental ideal $\quad I := \operatorname{Ker} \text{rank}$

$$GW(k) \supseteq I \supseteq I^2 \supseteq \ldots$$

$$K^M_{*} = \frac{\bigoplus_{i=0}^{\infty} \bigotimes_{j=1}^{i} (k^*)}{\langle a \otimes (1-a) \rangle}$$

Milnor K-theory groups

Milnor conjecture/
Thm of Voevodsky

$$1 \to \mathbb{Z}/2 \to \overline{k}^* \to \overline{k}^* \to 1$$

$$k^* \longrightarrow H^1_{et}(k, \mathbb{Z}/2)$$

$$I^n/I^{n+1} \xleftarrow{\cong} K^M_n(k) \otimes \mathbb{Z}/2 \xrightarrow{\cong} H^n_{et}(k, \mathbb{Z}/2)$$

$$(\langle 1\rangle - \langle a_1\rangle) \cdots (\langle 1\rangle - \langle a_n\rangle) \longleftarrow a_1 \otimes \cdots \otimes a_n$$

view maps $I^n \longrightarrow I^n/I^{n+1}$
as invariants on $GW(k)$

$n=0$: rank
$n=1$: discriminant
$n=2$: Hasse-Witt invariant
$n=3$: Arason invariant
$\vdots$

$B: V \times V \to k$
$\mathrm{disc}(B) = \det(B(v_i, v_j))$
$\{v_1, \ldots, v_n\}$ is a basis

Degree

Thm (Morel) $n \geq 2$

$$[(S^1)^{\wedge n} \wedge \mathbb{G}_m^{\wedge j}, (S^1)^{\wedge n} \wedge \mathbb{G}_m^{\wedge r}] \cong K^{MW}_{r-j}$$

e.g. $j=r=n$ $\quad [\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \cong GW(k)$

$k = \mathbb{R}$

$$\begin{tikzcd}
{[S^{2n}, S^{2n}]} \arrow[d, "\deg"] & {[\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}]} \arrow[l, "\mathbb{C}\text{-pts}"'] \arrow[r, "\mathbb{R}\text{-pts}"] \arrow[d, "\deg"] & {[S^n, S^n]} \arrow[d, "\deg"] \\
\mathbb{Z} & GW(k) \arrow[l, "\text{rank}"] \arrow[r, "\text{Signature}"'] & \mathbb{Z}
\end{tikzcd}$$

$K^{MW}_*(k)$ ~~the~~ Milnor-Witt K-theory

(Hopkins - Morel)

generators: $[a]$ $a \in k^*$ deg 1

η deg -1

relations:

$$\eta[a] = [a]\eta$$

$$[a][1-a] = 0 \quad \text{(Steinberg relation)}$$

$$[ab] = [a] + [b] + \eta[a][b]$$

$$\eta h = 0$$

$$GW(k) \cong K^{MW}_0(k)$$

$$\langle a \rangle \mapsto 1 + \eta[a]$$

$$h = \langle 1 \rangle + \langle -1 \rangle \mapsto h = 2 + \eta[-1]$$

$GW(k)$, $K^{MW}_*(k)$, $K^M_*(k)$ are global sections of sheaves.

Procedure for producing a sheaf K^{MW}_* from $K^{MW}_*(E)$ E finite type over k field

plus data

$v : E \longrightarrow \mathbb{Z} \cup \{\infty\}$ valuation

$\mathcal{O}_v = \{ e \in E \mid v(e) \geq 0 \}$

π uniformizer $v(\pi) = 1$

$k(v) := \mathcal{O}_v / \langle \pi \rangle$

$$\partial^\pi_v : K^{MW}_*(E) \longrightarrow K^{MW}_{*-1}(k(v))$$

$$\partial^\pi_v ([\pi][a_1] \cdots [a_n]) = [\bar{a}_1] \cdots [\bar{a}_n]$$

$a_i \in \mathcal{O}_v^*$

$$\partial^\pi_v ([a_0] \cdots [a_n]) = 0$$

$$K^{MW}_*(\mathrm{Spec}\,\mathcal{O}_v) = \ker \partial^{\pi}_v$$

$\rightsquigarrow$ K^{MW}_* is a sheaf

$$U \mapsto [S^0 \wedge U_+, \mathbb{G}_m^{\wedge *}]$$

$\rightsquigarrow$ transfers

$$\langle 1 \rangle = 1$$

$$\partial^{\pi}_v(n \cdot 1) = n\, \partial^{\pi}_v(1)$$