Enumerative geom: counts algebro-geom objects satisfying conditions

over $\mathbb{C}$

Goal: record information about fields of definition

Arithmetic Count of the lines on a Smooth Cubic surface

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joint with Jesse Kass

Def: Cubic surface is $X = \{(x,y,z) \mid F(x,y,z) = 0\}$

F is degree 3

Better: $X \subseteq \mathbb{P}^3 = \{[w,x,y,z]\}$

$[w,x,y,z] = [\lambda w, \lambda x, \lambda y, \lambda z]$

$\lambda \in k^*$

homog deg 3

$X = \{[w,x,y,z] \mid f[w,x,y,z) = 0\}$

<u>Theorem</u> (Salmon + Cayley 1849) Let X be a smooth, cubic surface over $\mathbb{C}$. Then X contains exactly 27 lines.

<u>Ex</u>: Fermat $\quad F(w,x,y,z) = w^3 + x^3 + y^3 + z^3$

$$L = \cancel{\{[S,T] \in \mathbb{P}^1}$$

$$\{[S,-S,T,-T] : [S,T] \in \mathbb{P}^1\}$$

$\lambda, w : \lambda^3 = w^3 = -1$

lines $\quad \{[S, \lambda S, T, wT] : [S,T] \in \mathbb{P}^1\}$

This produces: $\quad 3 \cdot 3 \cdot \dfrac{\binom{4}{2}}{2} = 27$

lines

modern proof:

$Gr(1,3)$ = Grassmannian parameterizing lines in $\mathbb{P}^3$

equivalatly parametrizing $W \subseteq \mathbb{C}^4$

$\dim W = 2$

Let $S \longrightarrow Gr(1,3)$ be tautological bundle

$$S_W := W$$

$$\mathrm{Sym}^3 S^* \longrightarrow Gr(1,3)$$

$\mathrm{Sym}^3 S^*{}_W$ = cubic polynomials on W, i.e. $\mathrm{Sym}^3 W^*$

F determines elt $\mathrm{Sym}^3(\mathbb{C}^4)^*$

$\Rightarrow F$ determines a section of $\mathrm{Sym}^3 S^*$ σ_F

by $\sigma_F(W) = F|_W$

$p \in M \qquad \sigma(p) = 0$

To Define: $\quad \deg_p \sigma \in \mathbb{Z}$

Here's how: choose local coords on M around p

There's a small ball around p with no other zeros

choose local trivialization of V compatible with relative orientation

Then σ can be identified with a function

$$\begin{array}{ccc} \sigma: \mathbb{R}^r & \longrightarrow & \mathbb{R}^r \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ 0 & \longmapsto & 0 \end{array}$$

$$\sigma(\overline{B_0(1)} - 0) \subset \mathbb{R}^r - 0$$

$$S^{r-1} = \partial B_0(1) \xrightarrow{\overline{\sigma}} \partial B_0(1) = S^{r-1}$$

$$x \longmapsto \frac{\sigma(x)}{|\sigma(x)|}$$

Note: the line $\mathbb{P}W$ corresponding to W is in X

$$\Longleftrightarrow$$

$$\sigma_F(W) = 0$$

want: count zeros of σ_F

Euler class: $V \longrightarrow M$ be a rank r $\mathbb{R}$-vector bundle on a dim r $\mathbb{R}$-mfld M

Assume V is oriented

Choose a section σ with only isolated zeros.

$$\deg : \underbrace{[S^{r-1}, S^{r-1}]}_{\text{homotopy classes of maps}} \longrightarrow \mathbb{Z}$$

Then $\deg_p \sigma := \deg(\overline{\sigma})$

Euler class

$$e(V) = \sum_{p:\ \sigma(p)=0} \deg_p \sigma$$

Fact: X smooth $\Rightarrow \deg_p \sigma = 1$

$\Rightarrow$ # lines on X = $e(\mathrm{Sym}^3 S^*)$

In particular, # lines is independent of X!

$$e(\mathrm{Sym}^3 S^*) = 27 \qquad \square$$

Q: What about cubic surfaces over $\mathbb{R}$?

Schläfli 19th century: X can have

3, 7, 15, or 27 real lines

Segre 1942 distinguished between hyperbolic and elliptic real lines on X

<u>Recall</u>: L real line $\quad L \cong \mathbb{P}^1_{\mathbb{R}}$

$$\mathrm{Aut}(L) \equiv \mathbb{P}GL_2\,\mathbb{R}$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ I & \longleftrightarrow & I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array}$$

$$z \mapsto \frac{az+b}{cz+d}$$

$$\mathrm{Fix}(I) = \{ z : cz^2 + (d-a)z + b = 0 \}$$

either consists of

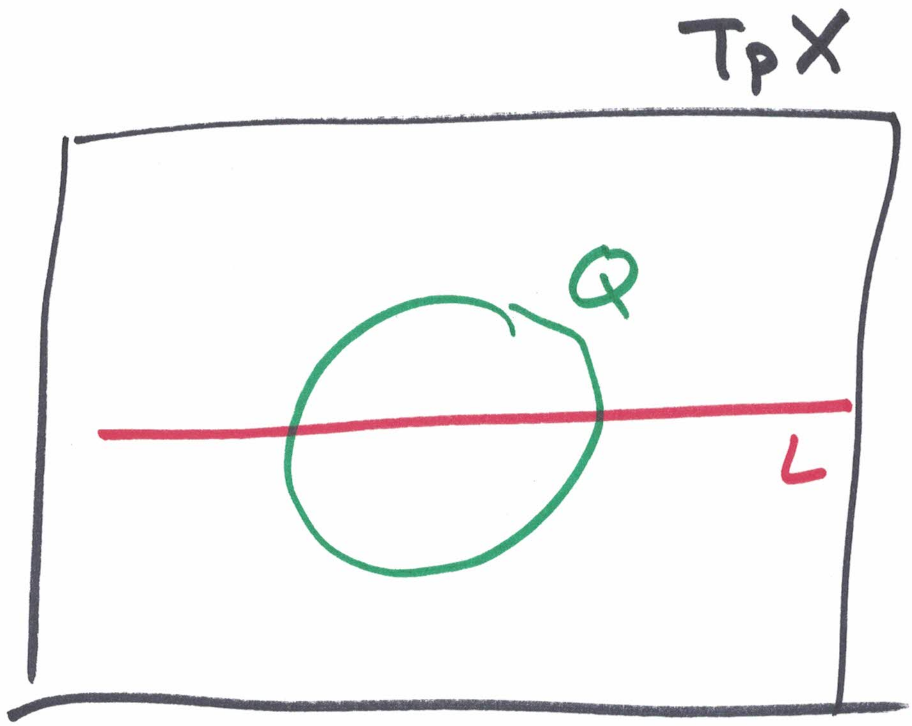
2 real points $\Longleftrightarrow$ I hyperbolic

a complex conj pair of pts $\Longleftrightarrow$ I elliptic

We associate an involution I to $L \subset X$ a real line on a real cubic surface.

$p \in L$

$$T_pX \cap X = L \cup Q$$



$Q \cap L = $ pts q s.t. $T_qX = T_pX$

$= \{p, p'\}$

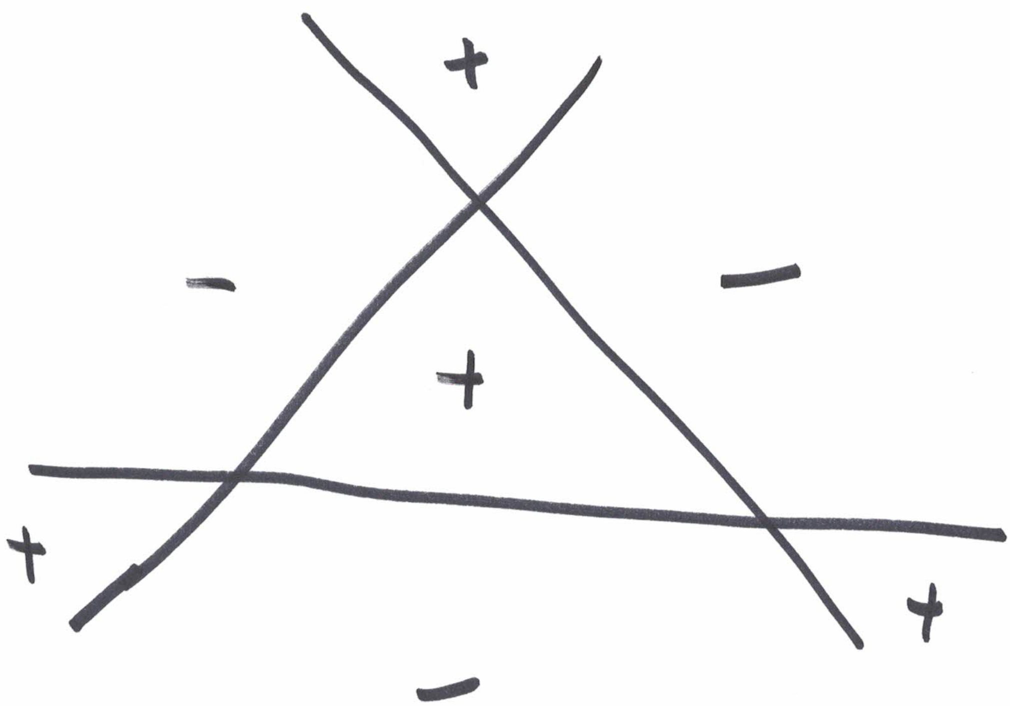
$$\boxed{I(p) = p'}$$

Def: L is elliptic / hyperbolic when I is.

Alternatively: Spin structures

Ex: Fermat cubic surface

$$x^3+y^3+z^3=-1$$



hyperbolic lines

Thm: (Segre + Okonek - Teleman
Finashin - Kharlamov
Benedetti - Silhol
Horev - Solomon)

$$\#\text{ hyperbolic lines} - \#\text{ elliptic lines} = 3$$

$\mathbb{A}^1$-homotopy theory Morel-Voevodsky
on Sm_k, k field

Morel deg: $[\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \to GW(k)$

$GW(k)$ = Grothendieck-Witt group
= group completion of semi-ring $\oplus, \otimes$ isomorphism classes of non-degenerate, symmetric bilinear forms $B: V \times V \to k$

finite dim'l k vector spaces

presentation:

generators: $\langle a \rangle \qquad a \in k^*$

$$\langle a \rangle : k \times k \longrightarrow k$$
$$(x, y) \longmapsto axy$$

relations: $\langle ab^2 \rangle = \langle a \rangle \qquad b \in k^*$

$$\langle a \rangle + \langle b \rangle = \langle a+b \rangle + \langle ab(a+b) \rangle$$

Ex: $GW(\mathbb{C}) \underset{\text{rank}}{\xrightarrow{\cong}} \mathbb{Z}$

$$B \longmapsto \dim V$$

Ex:
$$\begin{tikzcd} GW(\mathbb{R}) \arrow[r, "\text{signature} \times \text{rank}"] \arrow[dr, "\cong"'] & \mathbb{Z} \times \mathbb{Z} \\ & \mathbb{Z} \times \mathbb{Z} \arrow[u, phantom, "\subseteq"] \end{tikzcd}$$

Ex: $GW(\mathbb{F}_q) \underset{\cong}{\xrightarrow{\text{disc} \times \text{rank}}} \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2 \times \mathbb{Z}$

There is an Euler class

$$e(V) = \sum_{p : \sigma(p) = 0} \deg_p \sigma$$

k field char $\neq 2$

X smooth cubic surface $/ k$

line $L \subseteq X$ is a closed pt of $Gr(1,3)$

$$L = \{ [a,b,c,d]S + [a',b',c',d']T \\ [S,T] \in \mathbb{P}^1 \}$$

$$k(L) = k(a,b,c,d,a',b',c',d')$$

$$\mathbb{P}^1_{k(L)} \cong L \underset{\text{subscheme}}{\subseteq} X_{k(L)} \subseteq \mathbb{P}^3_{k(L)}$$

Given a line L on X,

obtain involution $I \in \mathrm{Aut}(L) \cong \mathbb{P}GL_2 k(L)$

$\mathrm{Fix}(I)$ is either 2 $k(L)$-pts

or

a conjugate pair of pts in $k(L)[\sqrt{D}]$ for $D \in k(L)^*/(k(L)^*)^2$

<u>Def:</u> $\mathrm{Type}(L) = \langle D \rangle \in GW(k(L))$

equiv: $D = ab - cd$ $\quad I = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\mathrm{Type}(L) = \langle -1 \rangle \deg I$

char $k \neq 2$

<u>Theorem</u> (Kass–W.) X smooth cubic surface

$$\sum_{\substack{\text{lines } L \\ \text{of } X}} \mathrm{Tr}_{k(L)/k}\, \mathrm{type}(L) = 15\langle 1 \rangle + 12\langle -1 \rangle$$

$$\mathrm{Tr}_{k(L)/k} : GW(k(L)) \to GW(k)$$

$$(B : V \times V \to k(L)) \mapsto$$

$$\begin{array}{ccc} V \times V & \xrightarrow{B} & k(L) \\ & & \downarrow \mathrm{Tr}_{k(L)} \\ & & k \end{array}$$

• $R = \mathbb{C}$ apply rank

$\#$ lines $= 27$

• $R = \mathbb{R}$ apply signature

$\#$ hyperbolic lines $-$ $\#$ elliptic lines $= 3$

<u>Cor</u>: $R = \mathbb{F}_q$

$$\left\{ \begin{array}{l} \#\text{ elliptic lines } L \\ \text{with } k(L) = \mathbb{F}_{q^{2n+1}} \end{array} \right\} + \left\{ \begin{array}{l} \#\text{ hyperbolic lines } L \\ \text{with } k(L) = \mathbb{F}_{q^{2n}} \end{array} \right\} \equiv 0 \pmod{2}$$