PROBLEMS ABOUT FORMAL GROUPS AND COHOMOLOGY THEORIES
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1. Formal Group Laws

In this document, a formal group law over a commutative ring R is always commutative and one-dimensional. Specifically, it is a power series $F(x, y) \in R[[x, y]]$ satisfying the following axioms:

- $F(x, 0) = x = F(0, x)$,
- $F(x, y) = F(y, x)$, and
- $F(x, F(y, z)) = F(F(x, y), z)$.

Sometimes we denote $F(x, y)$ by $x +_F y$. For a non-negative integer n, the n-series of F is defined inductively by $[0]_F x = 0$, and $[n+1]_F x = x +_F [n]_F x$.

(1) Show that for any formal group law $F(x, y) \in R[[x, y]]$, x has a formal inverse. In other words, there is a power series $i(x) \in R[[x]]$ such that $F(x, i(x)) = 0$. As a consequence, we can define $[-n]_F x = i([n]_F x)$.

(2) Check that the following define formal group laws:
   (a) $F(x, y) = x + y + uxy$, where u is any unit in R. Also show that this is isomorphic to the multiplicative formal group law over R.
   (b) $F(x, y) = \frac{x+y}{1+xy}$. *Hint:* Note that if $x = \tanh(u)$ and $y = \tanh(v)$, then $F(x, y) = \tanh(x + y)$.

(3) Suppose F, H are formal group laws over a complete local ring R, and we're given a morphism $f : F \to H$, i.e. $f(x) \in R[[x]]$ such that
$$f(F(x, y)) = H(f(x), f(y)).$$
Now let $\mathcal{C}$ be the category of complete local R-algebras and continuous maps between them; for $T \in \mathcal{C}$, denote by $\mathrm{Spf}(T) : \mathcal{C} \to \mathcal{S}et$ the functor $S \mapsto \mathrm{Hom}_{\mathcal{C}}(T, S)$.
   (a) Show that a formal group law F equips $\mathrm{Spf}\, R[[x]]$ with an abelian group structure. In other words, F determines a multiplication map $\mathrm{Spf}\, R[[x]] \times \mathrm{Spf}\, R[[x]] \to \mathrm{Spf}\, R[[x]]$ and a unit map $\mathrm{Spf}\, R \to \mathrm{Spf}\, R[[x]]$, making the usual diagrams commute. Denote the corresponding formal group by G_F.
   (b) Show that a morphism $f : F \to H$ as above corresponds exactly to a formal group homomorphism $G_F \to G_H$.

(4) Determine the p-series $[p]_{\mathbb{G}_m}(x)$ of the formal multiplicative group $\mathbb{G}_m$, and the height of $\mathbb{G}_m$.

(5) Using the p-series, show that if k is a field of characteristic p, there is no non-zero homomorphism from $\mathbb{G}_a$ to $\mathbb{G}_m$ over k.

(6) Let $g(x) \in R[[x]]$ be a power series of the form $g(x) = x + \sum_{i>1} b_i x^i$.
   (a) Show that there is a power series $g^{-1}(x) \in R[[x]]$ such that $g(g^{-1}(x)) = x = g^{-1}(g(x))$.
   (b) Show that $F_g(x, y) = g^{-1}(g(x) + g(y))$ is a formal group law.
   (c) If $R = \mathbb{Z}[e]/e^2$ and $g(x) = x + ex^n$, what is F_g?

(7) Let $R = \mathbb{Q}$, and let $g_n(x) = x + p^{-1}x^{p^n} + p^{-2}x^{p^{2n}} + \cdots$, and consider $F_n := F_{g_n}$ defined as in the previous exercise.
   (a) Assume you know that F_n is a formal group law defined ver $\mathbb{Z}$. (Can you prove that? – use Hazewinkel's functional equation lemma.) Calculate that $[p]_{F_n} \equiv x^{p^n} \mod p$. Conclude that the reductions Γ_n of F_n mod p for different n are all non-isomorphic.
   (b) Prove that $\phi(x) = ax$, for $a \in \bar{\mathbb{F}}_p$ is an endomorphism of Γ_n over $\bar{\mathbb{F}}_p$ if and only if $a \in \mathbb{F}_{p^n}$.
   (c) Prove that the endomorphism ring of Γ_n over $\mathbb{F}_p$ is the $\mathbb{Z}_p$-algebra generated by $S(x) = x^p$.
   (d) Prove that the endomorphism ring of Γ_n over $\bar{\mathbb{F}}_p$ is the $\mathbb{W}(\mathbb{F}_{p^n})$-algebra generated by $S(x)$.

(8) Let F be a formal group law over a torsion-free ring R. Show that the formal expression

$$f(x) = \int_0^x \frac{dt}{\frac{\partial F}{\partial y}(t, 0)} \in R_{\mathbb{Q}}[[x]]$$

satisfies the identity

$$f(F(x, y)) = f(x) + f(y),$$

so f is an isomorphism from F to $\mathbb{G}_a$ over $R_{\mathbb{Q}}$. In particular, any formal group law over a $\mathbb{Q}$-algebra is isomorphic to $\mathbb{G}_a$. *The power series $f(x)$ is called the logarithm of F and denoted* $\log_F(x)$.

(9) Suppose F is a formal group law over a torsion-free ring R. Show that the logarithm $\log_F(x)$ is the unique power series satisfying $\log_F [p]_F(x) = p \log_F x$.

(10) Let F be a formal group law over a torsion-free ring. Then its logarithm $\log_F(x)$ can be obtained as

$$\log_F(x) = \lim_{n\to\infty} p^{-n}[p^n]_F(x).$$

(11) Let F be a formal group law over R, and suppose the integer n is invertible in R. Show that the map $[n] : F \to F$ is an isomorphism.

(12) Suppose k is a perfect field of characteristic p, and suppose A is a complete local ring with maximal ideal $\mathfrak{m}$. The (p-typical) Witt vectors $\mathbb{W}(k)$ is a complete local ring with the following universal property: for any map of fields $i : k \to A/\mathfrak{m}$, there is a unique continuous map $\mathbb{W}(k) \to A$ which reduces to i modulo the maximal ideals. Show that this implies that the Lubin-Tate ring $E(k, \Gamma)$ classifying deformations of a formal group law Γ over k is a $\mathbb{W}(k)$-algebra.

(13) Try to prove directly that $\mathbb{Z}_p[\zeta]$, where ζ is a primitive $(p^n - 1)$-st root of unity, satisfies the above universal property to be the Witt vectors of $\mathbb{F}_{p^n}$.

(14) Let C be the elliptic curve over $\mathbb{F}_9$ defined by the equation $y^2 = x^3 - x$.
   (a) Show that the formal group law F_C of C has height 2, i.e. C is supersingular.

(b) Let $\tilde{C}$ be the elliptic curve over $\mathbb{W}(\mathbb{F}_9)[[u_1]]$ defined by the equation $y^2 = 4x^3 + u_1x^2 + 2x$. Show that the formal group law $F_{\tilde{C}}$ of $\tilde{C}$ is a universal deformation of the formal group law F_C of C.
(c) Suppose $(x, y) \to (\lambda^2 x, \lambda^3 y)$ defines an automorphism of C for some $\lambda \in \mathbb{F}_9^\times$. Show that λ has order 4, and conclude that $(\mathbb{F}_9^\times)^2$ is a subgroup of the automorphism group of C, and hence of the formal group law F_C.
(d) For $\lambda \in (\mathbb{F}_9^\times)^2$, compute the induced action on $\mathbb{W}(\mathbb{F}_9)[[u_1]]$.
(e) At the prime 2 instead of the 3, do a similar exercise for $y^2 + y = x^3$ over $\mathbb{F}_4$ and $y^2 + u_1xy + y = x^3$ over $\mathbb{W}(\mathbb{F}_4)[[u_1]]$.

2. Cohomology Theories and Formal Group Laws

Let $\mathcal{S}$ be the category of topological spaces, assumed to be compactly generated and weak Hausdorff. The assumption implies the existence of internal homs, aka mapping spaces. We study spaces up to homotopy, and in particular, any space is homotopy equivalent to a CW complex. Since things can be homotopic in multiple ways, and there are homotopies between homotopies etc, the correct way to encode all this is using model categories or infinity categories. Moreover, we really use the notion of a weak equivalence, i.e. an isomorphism on homotopy groups.

Denoting by $\mathcal{S}_*$ the category of pointed spaces, a (generalized, reduced) cohomology theory is a functor $\mathcal{S}_*^{op} \to \mathcal{A}b^{gr}$ to graded abelian groups, which is homotopy invariant, stable, exact, and additive. Examples include ordinary (singular) cohomology with arbitrary coefficients, complex and real K-theory, complex cobordism. Note that ordinary cohomology with commutative ring coefficients, as well as the other mentioned cohomology theories are multiplicative, i.e. the corresponding functors land in graded commutative rings.

(15) Write out what the axioms of a generalized cohomology theory mean.

Cohomology theories and natural transformations (aka cohomology operations) form a stable additive category. The category $\mathcal{S}p$ of spectra is a refinement of this, which allows for many natural constructions to be possible (eg. taking fibers or cofibers, localizations, etc.). At the end, spectra form a stable symmetric monoidal model category or ∞-category, whose tensor product is called the smash product (denoted $\wedge$), and the unit is the sphere spectrum S^0.

Suggested references: Adams's "Stable Homotopy and Generalized Homology" combined with the first sections of Bousfield-Friedlander's "Homotopy theory of Γ-spaces, spectra, and bisimplicial sets", or at the other end of the spectrum, Lurie's "Higher Algebra", with lots of options inbetween, a notable example being the appendix of Hill-Hopkins-Ravenel's "On the nonexistence of elements of Kervaire invariant one."

In the most naive definitions, a spectrum E is a sequence $\{E_n, \sigma_n\}_{n\in\mathbb{Z}}$ of pointed spaces E_n along with structure maps $\sigma_n : \Sigma E_n = E_n \wedge S^1 \to E_{n+1}$. A spectrum E defines a homology and cohomology theory for (pointed) spaces X:

$$E_nX = \lim_{k\to\infty} \pi_{n+k}(X \wedge E_k), \qquad E^n(X) = \lim_{k\to\infty} [\Sigma^k X, E_{n+k}].$$

(These are really the reduced cohomology theories; The unreduced versions are the values on X_+, the space X with a disjoint base point.)

For any pointed space X, its suspension spectrum $\Sigma^\infty X$ consists of $\Sigma^n X$, with identities as the structure maps. The sphere spectrum S^0 is $\Sigma^\infty(*_+)$; the homotopy groups of a spectrum E are defined as

$$\pi_n E = \lim_{k\to\infty} \pi_{n+k} E_k.$$

Formal group laws in homotopy theory appear when studying complex orientable cohomology theories.

Suppose $\xi \to X$ is a (complex) vector bundle; its Thom space is defined as the quotient $Th\xi = D(\xi)/S(\xi)$ of the unit disc bundle modulo the unit sphere bundle in ξ. This construction is natural (for maps of vector bundles), and the projection defines a map $p : Th\xi \to X$.

(16) What is the Thom space of a trivial vector bundle?

A multiplicative cohomology theory E is complex oriented if for any complex vector bundle $\xi \to X$ of (complex) rank n, there is a class $u_\xi \in E^{2n}(Th\xi)$ such that:

- For each point $x \in X$, the composite
$$E^{2n}(Th\xi) \to E^{2n}(Th\xi|_x) \cong E^{2n}(S^{2n}) \cong E^0(S^0)$$
sends u_ξ to 1;
- For any map $f : Y \to X$, we have $u_{f^*\xi} = f^* u_\xi$; and
- If η is another complex vector bundle on X, then $u_{\xi\oplus\eta} = u_\xi u_\eta$.

(17) Show that $p^*(-)u_\xi$ determines an isomorphism $E^k(X) \to E^{k+2n}(Th\xi)$. This is called a Thom isomorphism.

(18) The classifying space for complex line bundles is $\mathbb{C}P^\infty \cong BU(1)$. Let γ be the tautological line bundle on $\mathbb{C}P^\infty$. Show that the zero section $\mathbb{C}P^\infty \to Th\gamma$ is a homotopy equivalence.

(19) Let $i : S^2 \cong \mathbb{C}P^1 \to \mathbb{C}P^\infty$ be the map classifying the tautological line bundle on $\mathbb{C}P^1$. Show that a complex orientation for E gives a "first Chern" class $c_1^E \in E^2(\mathbb{C}P^\infty)$, such that $i^*(c_1^E) = 1 \in E^0 \cong E^2(S^2) = E^2(\mathbb{C}P^1)$.

(20) Prove the splitting principle, i.e. show that for any vector bundle $\xi \to X$, there is a space Y and a map $f : Y \to X$, such that $f^*\xi$ is a sum of line bundles.

(21) Conclude that a complex oriented cohomology theory has a natural theory of Chern classes for complex vector bundles, so that if $f_L : X \to \mathbb{C}P^\infty$ classifies a line bundle L on X, then $c_1^E(L) = f_L^*(c_1^E)$, and moreover, if $\xi \simeq L_1 \oplus \cdots \oplus L_n$, then $c_k(\xi)$ is the k-th symmetric elementary function on $c_1^E(L_1), \ldots c_1^E(L_n)$.

We denote by $c^E(\xi)$ the "total Chern class" $1 + c_1^E(\xi) + c_2^E(\xi) + \cdots \in E^*(X)$; then $c^E(\xi_1 \oplus \xi_2) = c^E(\xi_1)c^E(\xi_2)$.

Formal group laws appear when we address the question of what is the Chern class of a tensor product of bundles, or specifically for line bundles. There is a multiplication map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu} \mathbb{C}P^\infty$, such that if L_1, L_2 are line bundles over X, then the composite

$$X \xrightarrow{L_1 \times L_2} \mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu} \mathbb{C}P^\infty$$

classifies the tensor $L_1 \otimes L_2$. So, to understand what $c_1^E(L_1 \otimes L_2)$ is, we must understand what $E^*\mu$ is.

(22) Show that if E is complex oriented, there is an isomorphism $E^*(\mathbb{C}P^\infty) \cong E^*[[x_E]]$. You will need to understand the cell structure of $\mathbb{C}P^\infty$ and use the Atiyah-Hirzebruch spectral sequence. Then conclude that $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[x_E, y_E]]$.
(23) For an oriented spectrum E, let $F_E(x,y) \in E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[x,y]]$ be $\mu^*(x)$. Show that this is a formal group law.

The complex cobordism spectrum MU consists of the spaces $MU_{2n} = MU(n) = Th\xi_n$, where ξ_n is the tautological rank n-bundle on $BU(n)$, and $MU_{2n+1} = \Sigma MU_{2n}$. The structure maps are constructed from the inclusions $BU(n) \to BU(n+1)$.

(24) Show that MU is complex orientable.

In fact, MU is the universal oriented cohomology theory; orientations on a ring spectrum E correspond to ring maps $MU \to E$. In line with the above correspondence between orientations and formal group laws, Quillen showed that MU carries the universal formal group law, and so MU_* is the Lazard ring.

(25) Show that the complex K-theory spectrum is complex orientable.

3. Landweber exactness

A number of cohomology theories can be constructed from MU, using the following criterion due to Landweber. For each prime p and non-negative integer n, let v_n be the coefficient of x^{p^n} in $[p]_{F_{MU}}$, where v_0 is understood to be p. Suppose R is an MU_*-module. Suppose for each p, n, the map $v_n : R/(p, v_1, \dots, v_{n-1}) \to R/(p, v_1, \dots, v_{n-1})$ is injective, i.e. $(p, v_1, \dots)$ is a regular sequence. Then the functor

$$X \mapsto MU_*(X) \otimes_{MU_*} R$$

is a homology theory. Note, this does not require that R be a flat MU_*-module, but rather is related to a flatness condition on the moduli stack of formal groups.

(26) Show that the map $MU_* \to \mathbb{Z}$ classifying the additive formal group law satisfies he Landweber exactness criterion. (The homology theory one obtains this way is ordinary homology with integer coefficients.)
(27) Show that the map $MU_* \to \mathbb{Z}[u^{\pm 1}]$ classifying the multiplicative formal group law group $F(x,y) = x + y + uxy$ satisfies the Landweber exactness

criterion. (The homology theory one obtains this way is the one represented by the complex K-theory spectrum.)

(28) Let k be a perfect field of characteristic p, and Γ a formal group law over k. Show that the map $MU_* \to E(k,\Gamma)[u^{\pm 1}]$ classifying the universal deformation of Γ makes $E(k,\Gamma)[u^{\pm 1}]$ into an MU_*-algebra which satisfies Landweber's criterion.

(29) Let C be an elliptic curve over R. Its formal group is classified by a map $MU_* \to R[u^{\pm 1}]$. Under what conditions on C does this satisfy the Landweber criterion?

4. Group actions

Another way to get new cohomologies from old is in the presence of group actions. For a finite group G, a spectrum with a G-action is best defined in the ∞-categorical land: it corresponds to a functor (of ∞-categories) $X : BG \to \mathcal{S}p$, where BG the category with one object and whose endomorphisms are G. Then there are associated spectra X^{hG} (homotopy fixed points), X_{hG} (homotopy orbits), and X^{tG} (Tate spectrum), sitting in a cofiber sequence

$$X_{hG} \to X^{hG} \to X^{tG}.$$

Moreover, there are spectral sequences

$$\begin{aligned} H_s(G, \pi_t X) &\Rightarrow \pi_{t+s} X_{hG} \\ H^s(G, \pi_t X) &\Rightarrow \pi_{t-s} X_{hG}, \\ \hat{H}^s(G, \pi_t X) &\Rightarrow \pi_{t-s} X_{tG}, \end{aligned}$$

compatible with the cofiber sequence above.

(30) The group C_2 acts on the complex K-theory spectrum by complex conjugation, with homotopy fixed points being KO, the spectrum classifying real K-theory. Compute the associated homotopy fixed point spectral sequence, and then also the Tate and homotopy orbit spectral sequences.

(31) Fixing a perfect field k and formal group law Γ over k, let G be a finite group of automorphisms of Γ over k. Then G acts on the spectrum $\mathbf{E}(k,\Gamma)$ from (28) above. In specific examples, can you compute anything about this action?

(32) In the world of ∞-categories, we can also take homotopy fixed points of groups acting on categories. Let Vect be the category of finite dimensional vector spaces over some field; then Vect has an action by C_2 given by $V \mapsto V^\vee$. Show that the homotopy fixed points Vect^{hC_2} is the category of non-degenerate symmetric bilinear forms.