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Topological Hochschild homology in arithmetic geometry

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v.5 (27 Feb.) – Added a fun exercise to 2.18, some remarks on transitivity properties of HH, also a section on the mixed characteristic story.
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1 Introduction

This course is an introduction to Hochschild and cyclic homology and their topological variants, aimed mainly at readers whose background is in number theory and arithmetic geometry rather than algebraic topology and homotopy theory. Useful references to have at hand while reading these notes are the following:

- [29] The standard textbook presenting cyclic homology in a classical fashion; available free on SpringerLink if your institution subscribes.
- [28] Notes based on lectures given by Nikolaus on topological cyclic homology.
- [21] Notes from the Arbeitsgemeinschaft on topological cyclic homology at Oberwolfach, April 2018.
- [32] Nikolaus and Scholze's new approach to topological cyclic homology.
- [8] The article constructing "motivic" filtrations on all variants of (topological) cyclic homology and relating these theories to crystalline cohomology and p-adic Hodge theory.

Given a commutative base ring k and a k-algebra A, the Hochschild homology groups $HH_n(A/k)$, arising as the homology groups of the explicit Hochschild complex $HH(A/k)$, were introduced already in the 1950s and are closely related to algebraic differential forms: to be precise, if A is commutative then there exist maps $\Omega^n_{A/k} \to HH_n(A/k)$, which are even isomorphisms when A is smooth over k. In the 1980s it was realised by Connes and Feigin–Tsygan that the usual de Rham differential $d : \Omega^n_{A/k} \to \Omega^{n+1}_{A/k}$ was simply a shadow of the fact that the complex $HH(A/k)$ could be equipped with an action by the circle S^1, in a sense which we will take time to explain in the course. One may then study the homology, cohomology, and Tate cohomology of this S^1-action on $HH(A/k)$, giving rise respectively to cyclic homology, negative cyclic homology, and periodic cyclic homology; these fit together into a fibre sequence

$$HC(A/k)[1] \longrightarrow HC^-(A/k) \longrightarrow HP(A/k).$$

In §2 we will discuss these constructions from the classical point of view of explicit double complexes and the theory of mixed complexes, finally presenting a proof of the following classical description in the case of smooth algebras over a characteristic zero base ring:

Theorem 1.1 (Connes, Feigin–Tsygan)**.** *If A is a smooth k-algebra and $k \supseteq \mathbb{Q}$, then there are natural decompositions*

$$HP(A/k) = \prod_{n\in\mathbb{Z}} \Omega^{\bullet}_{A/k}[2n], \qquad HC^{-}(A/k) = \prod_{n\in\mathbb{Z}} \Omega^{\geq n}_{A/k}[2n]$$

In short, periodic and negative cyclic homology encode the de Rham cohomology of A and its Hodge filtration. The main interest of Connes was therefore to view these cohomology theories as replacements for de Rham cohomology, even if A is non-commutative. In this course we will rather be interested in the case in which A is commutative, but not necessarily smooth nor of characteristic zero. In particular our first goal will be to explain the following generalisation of the previous theorem to finite characteristic:

Theorem 1.2. *If A is a smooth k-algebra, where k is a field of characteristic $p > 0$, then $HP(A/k)$ and $HC^{-}(A/k)$ admit $\mathbb{Z}$-indexed, complete filtrations whose n^{th} graded pieces are given respectively by*

$$\Omega^{\bullet}_{A/k}[2n], \qquad \Omega^{\geq n}_{A/k}[2n].$$

In other words, the classical theorem of Connes and Feigin–Tsygan extends beyond characteristic zero, at the expense of replacing the product decomposition by a possibly non-split filtration.

The techniques which we will use to prove Theorem 1.2 in §3 are radically different to its characteristic zero predecessor. We will see that the cohomology theories involved can be determined by their behaviour on large $\mathbb{F}_p$-algebras, namely perfect rings and certain well-behaved quotients thereof (called quasiregular semiperfect rings). This leads us to the study of these cohomology theories on quasiregular semiperfect rings, and to the formalism of the quasisyntomic site.

The problem with Theorem 1.2 is that de Rham cohomology is not the most interesting cohomology theory in characteristic p; we would prefer to replace it by cristalline cohomology. However, it is manifestly impossible to build cristalline cohomology from $HH(A/k)$ via any $\mathbb{Z}$-linear construction: since $HH(A/k)$ is a complex of $\mathbb{F}_p$-vector spaces, so too will be anything constructed from it $\mathbb{Z}$-linearly. The trick is therefore to leave the $\mathbb{Z}$-linear world, i.e., $D(\mathbb{Z})$, by passing to the world of spectra Sp in the sense of homotopy theory; constructions in Sp are no longer $\mathbb{Z}$-linear, merely $\mathbb{S}$-linear where $\mathbb{S}$ is the sphere spectrum.

The idea of developing Hochschild and cyclic homology in this "brave new world" of ring spectra is due to Goodwillie and Wahldhausen. The subject has seen enormous progress in recent years, particularly thanks to pioneering work of Hesselholt and then the refoundation of the subject by Nikolaus–Scholze, whose framework we will adopt. To any ring A one may associated its topological Hochschild homology

$$THH(A)\text{“} := HH(A/\mathbb{S})\text{”} \in \mathrm{Sp},$$

which is once again equipped with an action by the circle S^1; passage to homology, cohomology, and Tate cohomology (which are no longer $\mathbb{Z}$-linear constructions!) then yields an analogous fibre sequence as classically

$$THH(A)_{hS^1}[1] \longrightarrow TC^{-}(A) \longrightarrow TP(A)$$

In §4 we attempt to gradually familiarise the reader with the concept of what it means for S^1 to act on an algebraic object like a chain complex, then categorify the construction of Hochschild homology in order to construct topological Hochschild homology.

Section 5 presents the main results concerning topological Hochschild homology of $\mathbb{F}_p$-algebras, after which there will be a brief discussion of the mixed characteristic situation.

Notation

Both chain and cochain complex appear in this theory, which is an easy source of confusion. We follow the standard convention of denoting the former by lower indexes (and often by leftwards arrows)

$$\cdots \leftarrow C_n \leftarrow C_{n+1} \leftarrow \cdots$$

and the latter by upper indexes

$$\cdots \to C^n \to C^{n+1} \to \cdots .$$

For shifting complexes we follow the convention $C[1]^n := C^{n+1}$, whence for chain complexes we have $C[1]_n = C_{n-1}$, i.e., given an abelian group A, the shift $A[n]$ is supported in cohomological degree $-n$ and in homological degree n; the differentials on $C[1]$ are given by the negatives of the differentials on C. Under this convention, the Eilenberg–Maclane functor takes the shift [1] to the suspension Σ. There are bound to be some mistakes involving signs of shifts...

2 Classical Hochschild and cyclic homology

In this section we give a classical presentation of Hochschild and cyclic homology and their variants, namely negative cyclic homology and periodic cyclic homology. We will define these in terms of the explicit double complexes which were introduced by Connes and Feigin–Tsygan and establish the periodicity and norm sequences relating the different theories. We then abstract the constructions by introducing the classical concepts of cyclic objects and mixed complexes, which will be helpful when discussing two important examples. In the final subsection we explain how the definitions should be modified when treating non-flat algebras, which is really just an excuse to recall the cotangent complex.

A standard reference for this material is chapters 1–5 of the book of Loday [29], also [39, §9] and [30]; we will give quite a streamlined presentation of the results we need, and consultations of these sources will probably be helpful for readers seeing the material for the first time.

2.1 Hochschild homology and differential forms

Let k be a commutative ring and A a flat k-algebra (the flatness hypothesis will be discussed and removed in §2.4). Then the associated *Hochschild complex*, denoted by $HH(A/k)$ or more classically by $HH^k(A)$, is the chain complex of k-modules

$$A \xleftarrow{b} A \otimes_k A \xleftarrow{b} A \otimes_k A \otimes_k A \xleftarrow{b} \cdots$$

(where A lies in degree 0) with boundary maps traditionally denoted by b given by

$$\begin{aligned} b : A^{\otimes_k n+1} \to A^{\otimes_k n}, \qquad a_0 \otimes \cdots \otimes a_n \mapsto\ & a_0a_1 \otimes \cdots \otimes a_n \\ & - a_0 \otimes a_1a_2 \otimes \cdots \otimes a_n \\ & + \cdots \\ & + (-1)^n a_0 \otimes \cdots \otimes a_{n-1}a_n \\ & + (-1)^{n+1} a_na_0 \otimes \cdots \otimes a_{n-1} \end{aligned}$$

For example, $b(a \otimes b) = ab - ba$ and $b(a \otimes b \otimes c) = ab \otimes c - a \otimes bc + ca \otimes b$. The homology groups of the Hochschild complex are denoted by $HH_n(A/k)$ are called the *Hochschild homology* of the k-algebra A.

Example 2.1 (Low degrees). (i) $n = 0$. The image of boundary map $A \otimes_k A \to A$ is precisely the k-submodule generated by elements of the form $ab - ba$, i.e., the commutator $[A, A]$ of A; so $HH_0(A/k) = A/[A, A]$. In particular, if A is commutative then $HH_0(A/k) = A$.

(ii) $n = 1$. Assume A is commutative. Then we have just seen that the first boundary map n the Hochschild complex is zero, so that

$$HH_1(A/k) = A \otimes_k A/\langle ab \otimes c - a \otimes bc + ca \otimes b : a, b, c \in A\rangle,$$

where $\langle\ \rangle$ denotes here the abelian group (or equivalently A-module) generated by the indicated elements. The relation by which we are quotienting is precisely the Leibniz rule, and so we see that there is an isomorphism of A-modules

$$HH_1(A/k) \xrightarrow{\simeq} \Omega^1_{A/k}, \qquad a \otimes b \mapsto a\, db.$$

Example 2.2 (The case $A = k$)**.** If $A = k$ then the boundary maps in the Hochschild complex are alternately zero and the identity, whence

$$HH_n(k/k) = \begin{cases} k & n = 0 \\ 0 & n > 0. \end{cases}$$

The higher Hochschild homology groups vanish more generally whenever A is a commutative étale k-algebra, as will follow from the Tor description of Remark 2.4.

Remark 2.3 (Simplicial perspective)**.** Recall that a simplicial object in a category consists of objects A_n, for $n \geq 0$, and morphisms

$$\begin{aligned} \text{"face maps"} \quad & d_i : A_n \to A_{n-1}, \quad i = 0, \dots, n, \\ \text{"degeneracy maps"} \quad & s_i : A_n \to A_{n+1}, \quad i = 0, \dots, n, \end{aligned}$$

satisfying the rules $d_i d_j = d_{j-1} d_i$ for $i < j$, and $s_i s_j = s_{j+1} s_i$ for $i \leq j$, and

$$d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ \text{id} & i = j \text{ or } j+1 \\ s_j d_{i-1} & i > j+1. \end{cases}$$

Diagramatically, we tend to forget about the degeneracy maps and draw such an object as

$$A_0 \leftleftarrows A_1 \Lleftarrow A_2 \Lleftarrow \cdots$$

A simplicial object is equivalently a functor to the category from Δ^{op}, where Δ is the simplicial category.

A simplicial object $A_\bullet$ in any additive category gives rise to a chain complex $A_0 \xleftarrow{d} A_1 \xleftarrow{d} A_2 \xleftarrow{d} \cdots$ where the boundary maps are given by the alternating sum of the face maps, namely $d := \sum_{i=0}^n (-1)^i d_i : A_n \to A_{n-1}$; this is known as the *un-normalised chain complex* associated to $A_\bullet$ (and, by a common abuse of notation, usually still denoted by $A_\bullet$).

In particular, to our flat k-algebra A we may associate a simplicial k-module defined by $A_n := A^{\otimes_k n+1}$ with face and degeneracy maps

$$d_i(a_0 \otimes \cdots \otimes a_n) := \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & i < n, \\ a_n a_0 \otimes \cdots \otimes a_{n-1} & i = n, \end{cases}$$

$$s_i(a_0 \otimes \cdots \otimes a_n) := a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n,$$

whose associated un-normalised chain complex is precisely the Hochschild complex $HH(A/k)$ as defined above. We will interchangeably use $HH(A/k)$ to denote both this simplicial k-module and the associated complex.

Remark 2.4 (Bar complex and Tor description)**.** Don't confuse $HH_n(A/k)$ with the *bar complex*

$$B_\bullet(A/k) := \qquad A \otimes_k A \xleftarrow{b'} A \otimes_k A \otimes_k A \xleftarrow{b'} \cdots$$

$$b' : A^{\otimes_k n} \to A^{\otimes_k n-1}, \qquad a_0 \otimes \cdots \otimes a_{n-1} \mapsto \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} a_i \otimes \cdots \otimes a_{n-1}$$

(i.e., same as the Hochschild complex except that the product of the first and last tensor factors does not appear), which can alternatively be described simplicially as

$$\begin{aligned} d_i(a_0 \otimes \cdots \otimes a_{n-1}) &:= a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ s_i(a_0 \otimes \cdots \otimes a_{n-1}) &:= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

In fact, the bar complex $B_\bullet(A/k)$ is acyclic except possibly in degree 0, with multiplication $b' = \mu : A \otimes_k A \to A$ defining a quasi-isomorphism $B_\bullet(A/k) \xrightarrow{\sim} A$; this follows formally from the existence of the "extra degeneracy" $s : A^{\otimes_k n} \to A^{\otimes_k n+1}$, $a_0 \otimes \cdots \otimes a_{n-1} \mapsto 1 \otimes a_0 \otimes \cdots \otimes a_{n-1}$, which serves as a contracting homotopy for $B_\bullet(A/k)$ as a simplicial object [29, 1.1.12].

The bar complex nevertheless plays a role in the theory in several ways. In particular, observe that it consists of left $A \otimes_k A^{\mathrm{op}}$-modules, under the rule $(f \otimes g)\cdot(a_0 \otimes \cdots \otimes a_{n-1}) := fa_0 \otimes a_1 \otimes \cdots \otimes a_{n-2} \otimes a_{n-1}g$, which are moreover flat since A is flat over k. Thus $B_\bullet(A/k) \xrightarrow{\sim} A$ serves as a resolution of A by flat $A \otimes_k A^{\mathrm{op}}$-modules (where we view A as a left $A \otimes_k A^{\mathrm{op}}$-module). But one checks directly that there is an isomorphism of complexes (even of simplicial objects)

$$A \otimes_{A \otimes_k A^{\mathrm{op}}} B_\bullet(A/k) \xrightarrow{\simeq} HH(A/k), \qquad a_0 \otimes (f \otimes a_1 \otimes \cdots \otimes a_n \otimes g) \mapsto ga_0 f \otimes a_1 \otimes \cdots \otimes a_n$$

(the fact that f and g change order is an inevitable consequence of the fact that the tensor product $A \otimes_{A \otimes_k A^{\mathrm{op}}}$ involves viewing A as a right $A \otimes_k A^{\mathrm{op}}$-module) whence we see that $HH(A/k)$ is a model for the derived tensor product $A \otimes^{\mathbb{L}}_{A \otimes_k A^{\mathrm{op}}} A$; in particular, $HH_*(A/k) \cong \mathrm{Tor}_*^{A \otimes_k A^{\mathrm{op}}}(A, A)$.

Remark 2.5 (The case A commutative)**.** In this remark we assume that A is a commutative k-algebra, which is our main case of interest. Then the face and degeneracy maps in $HH(A/k)$ are ring homomorphisms, which are even A-linear if we declare A to act on each $A^{\otimes_k n}$ by multiplication on the left-most tensor factor. In this way $HH(A/k)$ becomes a simplicial A-algebra and the boundary maps b are A-linear. It follows formally that the homology groups $HH_n(A/k)$ are A-modules, and that $HH_*(A/k) = \bigoplus_{n \geq 0} HH_n(A/k)$ has the structure of a graded-commutative A-algebra.

Similarly, $B_\bullet(A/k)$ is a simplicial $A \otimes_k A$-algebra and the identification $A \otimes_{A \otimes_k A^{\mathrm{op}}} B_\bullet(A/k) \cong HH(A/k)$ is compatible with multiplicative structure, whence $HH(A/k)$ models the simplicial commutative ring $A \otimes^{\mathbb{L}}_{A \otimes_k A} A$.

Remark 2.6 (Étale extensions)**.** Continue to suppose that A is a commutative, flat k-algebra, and let A' be an étale A-algebra. Then the canonical map $HH(A/k) \otimes_A A' \to HH_n(A'/k)$ is an isomorphism for any $n \geq 0$. This does not follow tautologically from the definition of Hochschild homology (even if A' is a localisation of A) but may be proved in several ways; here we present one approach, for another see [40].

Since $A \to A'$ is flat, it suffices to check that $HH(A/k) \otimes^{\mathbb{L}}_A A' \xrightarrow{\sim} HH(A'/k)$, which reduces via the Hochschild–Kostant–Rosenberg filtration of Proposition 2.28 to showing that $\mathbb{L}^i_{A/k} \otimes_A A' \xrightarrow{\sim} \mathbb{L}^i_{A'/k}$ for all $i \geq 0$ (where $\mathbb{L}$ is the cotangent complex, which will be introduced in §2.4). But this follows easily from the fundamental properties of the cotangent complex which will be presented in Proposition 2.27.

The cautious reader should be concerned about circular reasoning, given that the proof of Proposition 2.28 will Theorem 2.8, which in turn uses this remark, but in fact Proposition 2.28 only requires Theorem 2.8 in the case of polynomial algebras, whereas this remark will be used to extend from polynomial algebras to general smooth algebras.

Remark 2.7 (Transitivity)**.** Suppose that we are given a flat morphism of flat k-algebras $A \to B$, where A is assumed to be commutative. Then we claim that the canonical map of simplicial A-algebras is an isomorphism

$$HH(B/k) \otimes_{HH(A/k)} A \xrightarrow{\simeq} HH(B/A).$$

Here we use that A is commutative so that the adjunction $HH(A/k) \to HH_0(A/k)$ is a map to A. Since tensor products of simplicial rings are by definition computed degree-wise, the claimed isomorphism is simply the assertion that

$$(B \otimes_k \cdots \otimes_k B) \otimes_{A \otimes_k \cdots \otimes_k A, \mu} A \xrightarrow{\simeq} B \otimes_A \cdots \otimes_A B$$

for any tensor power, where we base change along the multiplication map μ. But this is obvious: the universal property of the right side yields the inverse map.

We finish this foundational subsection by discussing the case of smooth algebras. Assuming for a moment that A is any commutative, flat k-algebra, then the identification $\Omega^1_{A/k} \xrightarrow{\simeq} HH_1(A/k)$ of Example 2.1(i) induces a map of graded algebras $\Omega^*_{A/k} \to HH_*(A/k)$, simply because $HH_*(A/k)$ is a graded-commutative A-algebra by Remark 2.5. By explicitly examining the definition of the product structure on the homotopy groups of a simplicial commutative rings (known as the *shuffle product*), one can check that this map of graded algebras is given in each degree by the maps

$$\varepsilon_n : \Omega^n_{A/k} \longrightarrow HH_n(A/k), \qquad a\, db_1 \wedge \cdots \wedge db_n \mapsto \sum_{\sigma \in \mathrm{Sym}_n} (-1)^{\mathrm{sign}(\sigma)} a \otimes b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(n)}$$

for each $n \geq 0$ (however, we will try to avoid using this explicit formula). These are often known as the *anti-symmetrisation maps.*

We may now present the fundamental description of Hochschild homology of smooth algebras:

Theorem 2.8 (Hochschild–Kostant–Rosenberg Theorem 1962)**.** *If A is a smooth k-algebra, then the antisymmetrisation maps $\Omega^n_{A/k} \to HH_n(A/k)$ are isomorphisms of A-modules for all $n \geq 0$.*

Proof. The goal is to establish that the morphism of graded-commutative rings $\Omega^*_{A/k} \to HH_*(A/k)$, which is already known to be an isomorphism in degrees ≤ 1, is in fact an isomorphism in all degrees. It is therefore equivalent to prove that the canonical map $\bigwedge^*_A HH_1(A/k) \to HH_*(A/k)$ is an isomorphism; by now appealing to the Tor description of Hochschild homology from Remarks 2.4 and 2.5, we see that it is equivalent to check that the canonical map $\bigwedge^*_A \mathrm{Tor}_1^{A\otimes_k A}(A, A) \to \mathrm{Tor}_*^{A\otimes_k A}(A, A)$ is an isomorphism, i.e., that the Tor algebra $\mathrm{Tor}_*^{A\otimes_k A}(A, A)$ is the free graded-commutative A-algebra generated in degree 1 by $\mathrm{Tor}_1^{A\otimes_k A}(A, A) = I/I^2$, where $I = \mathrm{Ker}(A \otimes_k A \xrightarrow{\mu} A)$. This is well-known to be true if I is generated by a regular sequence (just compute the graded Tor ring using a Koszul complex).

We have completed the proof whenever $\mathrm{Ker}\,\mu$ is generated by a regular sequence, e.g., if A is a polynomial k-algebra.

But A is smooth over k, so by arguing Zariski locally on Spec A (here we use that $HH_n(A/k)\otimes_A A[\frac{1}{f}] = HH_n(A[\frac{1}{f}]/k)$ for all $f \in A$, thanks to Remark 2.6) we may reduce to the case that A is étale over a polynomial algebra; the result for A then follows by the polynomial case and another use of Remark 2.6. □

Remark 2.9 (Detecting smoothness with *HH*)**.** The HKR Theorem admits various converses; for example, if A is a non-smooth, but finitely generated, algebra over a field k, then $HH_n(A/k)$ is non-zero for infinitely many $n \geq 0$ [3].

Question 2.10. Which $\mathbb{E}_\infty$-rings A have the property that $THH_*(A) = \bigwedge^*_A THH_1(A)$? This might be known, I have not yet asked any experts.

2.2 Cyclic homology and its variants; cyclic objects and mixed complexes

The goal of this subsection is first to define classical cyclic homology and its variants, namely negative and periodic cyclic homology; then we will axiomatise the formalism by summarising Connes' theory of cyclic objects and Kassel–Burghelea's mixed complexes.

For the moment we continue to let A be a flat algebra over a commutative base ring k. The starting point for cyclic homology is the observation that each tensor power $A^{\otimes_k n+1}$ in the Hochschild complex carries a k-linear action by the cyclic group $\mathbb{Z}/n+1$ via permutation, i.e., the generator $t_n = 1 \in \mathbb{Z}/n+1$ acts[1] as

$$t_n \cdot a_0 \otimes \cdots \otimes a_n := a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

Following classical notation, we also introduce the k-linear maps

$$\text{“norm”} \quad N := \sum_{i=0}^{n} ((-1)^n t_n)^i : A^{\otimes_k n+1} \to A^{\otimes_k n+1},$$

$$\text{“extra degeneracy”} \quad s : A^{\otimes_k n} \to A^{\otimes_k n+1}, \qquad a_0 \otimes \cdots \otimes a_{n-1} \mapsto 1 \otimes a_0 \otimes \cdots \otimes a_{n-1},$$

and the composition

$$\text{“Connes' (boundary) operator”} \quad B := (1 - (-1)^n t_n)sN : A^{\otimes_k n} \longrightarrow A^{\otimes_k n+1}.$$

[1] Compared to the first version of these notes, the sign convention has changed.

When n is clear from the context one tends to write simply t instead of t_n to simplify notation; with this in mind we will adopt the (non-standard) convention of writing $\pm t := (-1)^n t_n$ so that, for example, $N = \sum_{i=0}^{n} (\pm t)^i$ and $B = (1 - \pm t)sN$.

Lemma 2.11. *The above operators satisfy the following identities:*

(i) $(1 - \pm t)b' = n(1 - \pm t)$ *and* $b'N = Nb$.

(ii) $sb' + b's = 1$, $B^2 = 0$, *and* $Bb = -bB$.

Proof. We leave these to the reader as an exercise; see [29, Lem. 2.1.1]. ☐

Remark 2.12 (Connes' operator vs. the de Rham differential)**.** Connes' boundary operator clearly induces k-linear maps $B : HH_n(A/k) \to HH_{n+1}(A/k)$ for $n \geq 0$. When A is commutative, as we assume in the rest of the remark, then this turns the graded algebra $HH_*(A/k)$ into a commutative differential graded k-algebra [To do: insert proof of this.] Since $HH_0(A/k) = A$, the universal property of $\Omega^*_{A/k}$ (namely, it is the initial cdg k-algebra which is A in degree 0) induces a map of differential graded k-algebras $\Omega^*_{A/k} \to HH_*(A/k)$. However this is nothing other than the anti-symmetrisation map constructed before Theorem 2.8, since the map

$$B : A = HH_0(A/k) \longrightarrow \Omega^1_{A/k} = HH_1(A/k)$$

is easily seen to be precisely the de Rham differential (the reader should check).

In particular, it follows that the following diagrams commute

$$\begin{array}{ccc} HH_n(A/k) & \xrightarrow{B} & HH_{n+1}(A/k) \\ \uparrow{\scriptstyle \varepsilon_n} & & \uparrow{\scriptstyle \varepsilon_{n+1}} \\ \Omega^n_{A/k} & \xrightarrow[d]{} & \Omega^{n+1}_{A/k} \end{array}$$

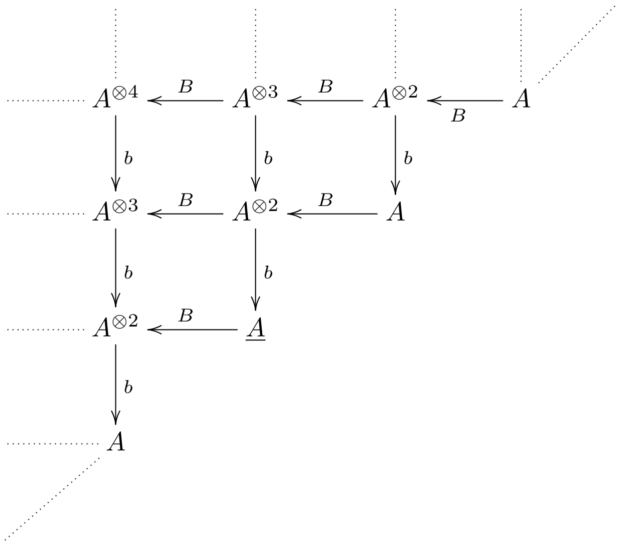
i.e., B is a refinement of the de Rham differential. Part of Connes' motivation when developing cyclic homology was to consider B as a generalisation of the de Rham differential even when A was non-commutative.

The classical double complex approach to defining cyclic homology and its variants are based on the following (anticommuting) bicomplex of k-modules, which is horizontally 2-periodic

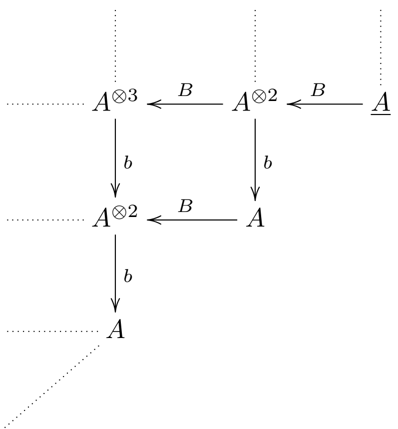
$$\begin{array}{ccccccccccccc} & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\ \cdots & & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-\pm t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-\pm t} & A^{\otimes 3} & & \cdots \\ & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\ \cdots & & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-\pm t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-\pm t} & A^{\otimes 2} & & \cdots \\ & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\ \cdots & & A & \xleftarrow{N} & \underline{A} & \xleftarrow{1-\pm t} & A & \xleftarrow{N} & A & \xleftarrow{1-\pm t} & A & & \cdots \end{array}$$

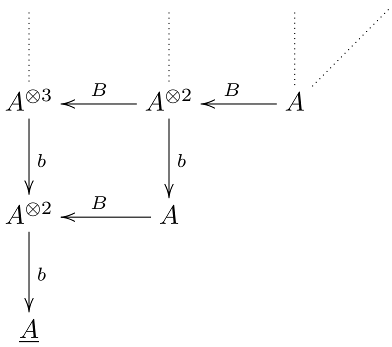
(the term in degree $(0,0)$ is underlined, a convention we will follow throughout this subsection), i.e., the even columns are the Hochschild complex $HH(A/k)$ and the odd columns are the augmented bar complex $B_\bullet(A/k) \to A$ from Remark 2.4. Restricting this bicomplex to the right half plane $x \geq 0$ gives a first quadrant bicomplex whose totalisation we define to be the *cyclic homology* $HC(A/k)$. Similarly, restricting to the left half plane $x \leq 0$ gives a second quadrant bicomplex whose product totalisation is defined to be *negative cyclic homology* $HC^-(A/k)$ (also denoted by $HN(A/k)$ in the literature). Finally, the product totalisation of the entire 2-periodic bicomplex is *periodic cyclic homology* $HP(A/k)$.

However, these double complexes can be simplified by recalling from Remark 2.4 that the odd columns in the above bicomplex are all acyclic, with contracting homotopy given by the extra degeneracy s; this allows us formally to remove all odd columns from the bicomplex by mapping between adjacent even columns via $(1 - \pm t)sN = B$. More precisely, it follows that $HP(A/k)$ is quasi-isomorphic to its subcomplex given by totalising the *BP-bicomplex*



Similarly, $HC^{-}(A/k)$ and $HC(A/k)$ are quasi-isomorphic to their subcomplexes given respectively by totalising the restriction of the BP-bicomplex to $x \leq 0$ or $x \geq 0$, which we call the *BC^{-}-bicomplex* and *BC-bicomplex*





For some further details on this process of discarding all all columns of the original bicomplexes, we refer the reader to [29, §2.1.7] or [30, Prop. 1.5].

Over the next sequence of remarks we explain various formal properties of cyclic homology and its variants, all of which follow by elementary examination of the double complexes defining them.

Remark 2.13 (Periodicity sequence for HC)**.** Here we explain the periodicity sequence through which cyclic homology is built from successive shifts of Hochschild homology.

Viewing $HC(A/k)$ as the totalisation of the latter BC-bicomplex, the inclusion of the zero-th column defines an inclusion $I : HH(A/k) \hookrightarrow HC(A/k)$; the cokernel of I is the totalisation of what remains after removing the first column from the BC-bicomplex, which is simply another copy of the BC-bicomplex, but shifted by bidegree $(1, 1)$. This gives a short exact sequence of complexes

$$0 \longrightarrow HH(A/k) \xrightarrow{I} HC(A/k) \xrightarrow{S} HC(A/k)[2] \longrightarrow 0$$

with associated boundary map $B : HC(A/k)[1] \to HH(A/k)$. The projection map $S : HC(A/k) \to HC(A/k)[2]$ from removing the zero-th column is known as *Connes' periodicity operator*.

In other words, $\mathrm{Fil}^i := \mathrm{Ker}(S^i : HC(A/k) \to HC(A/k)[2i])$ defines an exhaustive, increasing, $\mathbb{N}$-indexed filtration on $HC(A/k)$ whose i^{th} graded piece is $HH(A/k)[2i]$ for each $i \geq 0$. For explanation of our adjectives concerning filtrations, see Remark 2.29

Remark 2.14 (Periodicity sequences for HC^-)**.** Similarly to Remark 2.14, projecting the BC^--bicomplex to its zero-th column defines a short exact sequence of complexes

$$0 \longrightarrow HC^-(A/k)[-2] \xrightarrow{S} HC^-(A/k) \xrightarrow{h} HH(A/k) \longrightarrow 0,$$

and the images of all powers of S define a complete, descending, $\mathbb{N}$-indexed filtration on $HC^-(A/k)$ with graded pieces $HH(A/k)$[-2i], $i \geq 0$.

Remark 2.15 (Norm sequence)**.** Continuing the theme of Remarks 2.13 and 2.14, the inclusion I of the BC^--bicomplex into the BP-bicomplex has complement given by a shift of the BC-bicomplex, and there are projections h from the BC^--bicomplex to the BC-bicomplex, and from the BP-bicomplex to the BC-bicomplex. Using the sequences of the previous remarks, the summary is a commutative diagram of short exact sequences as follows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & HC^-(A/k)[-2] & = & HC^-(A/k)[-2] & & & & \\
 & & \downarrow S & & \downarrow IS & & & & \\
0 & \longrightarrow & HC^-(A/k) & \xrightarrow{I} & HP(A/k) & \xrightarrow{Sh} & HC(A/k)[2] & \longrightarrow & 0 \\
 & & \downarrow h & & \downarrow h & & \| & & \\
0 & \longrightarrow & HH(A/k) & \xrightarrow[I]{} & HC(A/k) & \xrightarrow[S]{} & HC(A/k)[2] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The connecting map $HC(A/k)[1] \to HC^-(A/k)$ associated to the short exact sequence in the middle is represented by the norm map N (this is easier to make precise using the original 2-periodic (b, b')-bicomplexes).

Remark 2.16 (Periodicity of HP)**.** The restriction of the BP-bicomplex to any right half plane $x \geq -n$ is of course a copy of the BC-bicomplex shifted by bidegree $(-n, -n)$. Taking the inverse limit as $n \to \infty$ shows that

$$HP(A/k) \simeq \mathrm{Rlim}_{n\to\infty} HC(A/k)[2n],$$

where the transition maps are shifts of the periodicity operator $S : HC(A/k) \to HC(A/k)[2]$. In other words, periodic cyclic homology is constructed by forcing the periodicty operator to be a quasi-isomorphism, and in particular it is 2-periodic: $S : HP(A/k) \xrightarrow{\sim} HP(A/k)[2]$.

Next we axiomatise the essential properties of the Hochschild complex which were used in the above constructions. However, we suggest that the reader first jumps ahead to the examples of §2.3 to see why this axiomatisation is useful.

Definition 2.17 (Connes [14])**.** A *cyclic object* in a category $\mathcal{C}$ is a simplicial object $X_\bullet$ in $\mathcal{C}$ such that each X_n is equipped with an automorphism t_n of order $n+1$, and these are required to satisfy:

$$t_n d_i = \begin{cases} d_{i-1} t_{n-1} & 1 \leq i \leq n \\ d_n & i = 0 \end{cases} \qquad t_n s_i = \begin{cases} d_{i-1} s_{n-1} & 1 \leq i \leq n \\ s_n t_{n+1}^2 & i = 0. \end{cases}$$

Equivalently, a cyclic object is a functor $\Lambda^{\mathrm{op}} \to \mathcal{C}$, where Λ is Connes' *cyclic category*, characterised as follows:

- Λ contains Δ as a full subcategory having the same objects;
- $\mathrm{Aut}_\Lambda([n]) = \mathbb{Z}/n+1$ for each $n \geq 0$ (whereas $\mathrm{Aut}_\Delta([n]) = 1$);
- The above relations between $t_n \in \mathbb{Z}/n+1$ and the face and degeneracy maps hold.

Checking that there really exists such a category Λ characterised by these properties requires some mild combinatorics which we omit; see [29, §6.1].

Example 2.18. The reader will easily observe that the simplicial k-module $HH(A/k)$ is indeed a cyclic object in k-modules, with the automorphisms t_n as defined at the start of the subsection.

Assume now that A is commutative, whence $HH(A/k)$ is even a cyclic object in k-algebras. In fact, as an exercise prove the following universal characterisation of the $HH(A/k)$: it is initial among the collection of pairs $(X_\bullet, A \to X_\bullet)$, where $X_\bullet$ is a cyclic object in k-algebras and $A \to X_\bullet$ is a morphism of simplicial k-algebras (i.e., the underlying simplicial object of $X_\bullet$ is given the structure of a simplicial A-module).

When viewed as a complex, the essential data on $HH(A/k)$ was instead as follows:

Definition 2.19 (Burghelea [13], Kassel [25])**.** A *mixed complex*, or *algebraic S^1-complex*, $(A_\bullet, b, B)$ over k is the data of a chain complex of k-modules

$$A_\bullet = \qquad \cdots \xleftarrow{b} A_n \xleftarrow{b} A_{n+1} \xleftarrow{b} \cdots$$

supported in homological degrees ≥ 0, together with maps of k-modules $B : A_n \to A_{n+1}$ satisfying both $Bb = -bB$ (i.e., B is a morphism of complexes $A \to A[1]$) and $B^2 = 0$.

Given a mixed complex $(A_\bullet, b, B)$, its *periodic cyclic homology* $HP(A_\bullet, b, B)$, *negative cyclic homology* $HC^-(A_\bullet, b, B)$, and *cyclic homology* $HC(A_\bullet, b, B)$, are the product totalisations of the following bicomplex:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
\cdots & & A_3 & \xleftarrow{B} & A_2 & \xleftarrow{B} & A_1 & \xleftarrow{B} & A_0 \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
\cdots & & A_0 & \xleftarrow{B} & A_1 & \xleftarrow{B} & A_0 & & \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & & & \\
\cdots & & A_1 & \xleftarrow{B} & \underline{A_0} & & & & \\
 & & \downarrow{\scriptstyle b} & & & & & & \\
\cdots & & A_0 & & & & & & \\
 & \swarrow & & & & & & &
\end{array}$$

resp. its restriction to the left half plane $x \leq 0$, resp. its restriction to the right half plane $x \geq 0$.

Example 2.20. (i) The Hochschild complex $HH(A/k)$ is a mixed complex with respect to Connes' operator B, by Lemma 2.11; in this case Definition 2.19 of course reduces to the earlier definitions of cyclic, negative cyclic, and periodic cyclic homology.

(ii) Given any cochain complex $\cdots \xrightarrow{B} C^n \xrightarrow{B} C^{n+1} \xrightarrow{B} \cdots$ supported in cohomological degrees ≥ 0, we can form a mixed complex $(C_\bullet, 0, B)$ by setting $C_n := C^n$ and declaring the b-differentials to be zero. Then the double complex of Definition 2.19 is a product of infinitely many copies of shifts of the cochain complex $(C^\bullet, B)$; truncating to the left or right half plane involves naively truncating this cochain complex. Taking care of the indexing, one sees that

$$HC^-(C_\bullet, 0, B) = \prod_{i\in\mathbb{Z}} C^{\geq i}[2i], \qquad HP(C_\bullet, 0, B) = \prod_{i\in\mathbb{Z}} C^\bullet[2i], \qquad HC(C_\bullet, 0, B) = \prod_{i\geq 0} C^{\leq i}[2i].$$

Of particular interest will be the mixed complex $(\Omega^\bullet_{A/k}, 0, d)$, where $\Omega^\bullet_{A/k}$ is the usual de Rham complex of a commutative k-algebra A.

(Warning: there is an unfortunate conflict of notation here: usually if we view a cochain complex $C^\bullet$ as a chain complex then it implicitly means that $C_n := C^{-n}$ with the same differential as in $C^\bullet$; but mixed complexes will appear sufficiently rarely that this conflict should not cause any problems.)

(iii) At the other extreme to example (ii), one can also consider mixed complexes $(A_\bullet, b, 0)$ in which the B-differential is zero. Then the double complex of Definition 2.19 is a product of infinitely many copies of shifts of the chain complex $A_\bullet$, and one sees that

$$HC^-(A_\bullet, b, 0) = \prod_{i\leq 0} A_\bullet[2i], \qquad HP(A_\bullet, b, 0) = \prod_{i\in\mathbb{Z}} A_\bullet[2i], \qquad HC(A_\bullet, b, B) = \prod_{i\geq 0} A_\bullet[2i].$$

Remark 2.21. Remarks 2.13–2.16 extend verbatim to arbitrary mixed complexes, from which one obtains the following sort of consequence (the reader should check if uncertain): given a morphism of mixed complexes $(A_\bullet, b, B) \to (A'_\bullet, b', B')$ (the notion should be clear) such that $A_\bullet \to A'_\bullet$ is a quasi-isomorphism of chain complexes, then $HC(A_\bullet, b, B) \to HC(A'_\bullet, b', B')$ is also a quasi-isomorphism, and similarly for HC^- and HP.

Now let $A_\bullet$ be a cyclic object in k-modules. Then we claim that its un-normalised chain complex

$$A_\bullet := \quad A_0 \xleftarrow{b} A_1 \xleftarrow{b} \cdots, \qquad b = \sum_{i=0}^{n} d_i$$

admits the structure of a mixed complex. Indeed, defining the "extra degeneracy" $s : A_{n-1} \to A_n$ to be $s := t_n s_n$, we may then define the norm N and boundary operator B exactly as at the start of the subsection and repeat the arguments of Lemma 2.11 verbatim to see that $(A_\bullet, b, B)$ is a mixed complex.

There is moreover an analogue of the bar complex in this generality, namely the chain complex

$$A_0 \xleftarrow{b'} A_1 \xleftarrow{b'} \cdots \qquad b' = \sum_{i=0}^{n-1} d_i,$$

which is again acyclic since s serves as a contracting homotopy. Just as for Hochschild homology we may therefore form the bicomplex

$$\begin{array}{ccccccccccccc}
& & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\cdots & & A_2 & \xleftarrow{N} & A_2 & \xleftarrow{1-\pm t} & A_2 & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-\pm t} & A_2 & & \cdots \\
& & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
\cdots & & A_1 & \xleftarrow{N} & A_1 & \xleftarrow{1-\pm t} & A_1 & \xleftarrow{N} & A_1 & \xleftarrow{1-\pm t} & A_2 & & \cdots \\
& & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
\cdots & & A_0 & \xleftarrow{N} & A_0 & \xleftarrow{1-\pm t} & \underline{A_0} & \xleftarrow{N} & A_0 & \xleftarrow{1-\pm t} & A_0 & & \cdots
\end{array}$$

and argue that $HP(A_\bullet, b, B)$, $HC^-(A_\bullet, b, B)$, and $HC(A_\bullet, b, B)$ could instead have been defined (up to quasi-isomorphism) as the totalisation of this bicomplex, resp. its truncation to $x \leq 0$, resp. to $x \geq 0$.

We will return to cyclic objects and mixed complexes, and their relation, from a more highbrow point of view in §4.

2.3 Two examples: smooth algebras and group algebras

2.3.1 Smooth algebras in characteristic zero

The main classical example in cyclic homology which will be important to us (albeit only as motivation) is the case of smooth algebras in characteristic zero, when it can be described in terms of de Rham cohomology. For any commutative k-algebra A, we denote as usual by $\Omega^\bullet_{A/k}$ its de Rham complex and by

$$\Omega^{\le i}_{A/k} := \qquad A \xrightarrow{d} \cdots \xrightarrow{d} \Omega^i_{A/k} \longrightarrow 0 \longrightarrow \cdots$$

its naive truncations; similarly define $\Omega^{\ge i}_{A/k}$, so that $0 \to \Omega^{\ge i}_{A/k} \to \Omega^\bullet_{A/k} \to \Omega^{\le i-1}_{A/k} \to 0$ is a short exact sequence of complexes.

Theorem 2.22 (Connes, Loday–Quillen, Feigin–Tsygan 1980s)**.** *Let k be a commutative base ring containing $\mathbb{Q}$ and A a smooth k-algebra. Then there are natural equivalences*

$$HC^-(A/k) \simeq \prod_{i\in\mathbb{Z}} \Omega^{\ge i}_{A/k}[2i], \qquad HP(A/k) \simeq \prod_{i\in\mathbb{Z}} \Omega^{\bullet}_{A/k}[2i], \qquad HC(A/k) \simeq \prod_{i\ge 0} \Omega^{\le i}_{A/k}[2i].$$

Proof. We define maps

$$\pi : A^{\otimes_k n+1} \longrightarrow \Omega^n_{A/k}, \qquad a\otimes b_1 \otimes \cdots \otimes b_n \mapsto a\, db_1 \wedge \cdots \wedge db_n$$

and leave it to the reader to check directly that the following diagrams commute:

$$\begin{array}{ccc} A^{\otimes_k n} & \xleftarrow{b} & A^{\otimes_k n+1} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \Omega^{n-1}_{A/k} & \xleftarrow[0]{} & \Omega^n_{A/k} \end{array} \qquad \begin{array}{ccc} A^{\otimes_k n} & \xrightarrow{B} & A^{\otimes_k n+1} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \Omega^{n-1}_{A/k} & \xrightarrow[(n+1)d]{} & \Omega^n_{A/k} \end{array}$$

In other words, the maps π define a morphism of mixed complexes from $HH(A/k)$ to $(\Omega^\bullet_{A/k}, 0, (\bullet{+}1)d) :=$ the mixed complex with zero b-differential associated to the cochain complex

$$A \xrightarrow{2d} \Omega^1_{A/k} \xrightarrow{3d} \Omega^2_{A/k} \xrightarrow{4d} \cdots .$$

We also leave it to the reader to check that the composition $\Omega^n_{A/k} \xrightarrow{\varepsilon_n} HH_n(A/k) \xrightarrow{H_n(\pi)} \Omega^n_{A/k}$ is simply multiplication by $n!$, where ε_n is the anti-symmetrisation map which appeared in Theorem 2.8.

Now we use the hypotheses on A. Since it is smooth over k, Theorem 2.8 implies that ε_n is an isomorphism; since $k \supseteq \mathbb{Q}$, it follows that $H_n(\pi)$ is also an isomorphism. Replacing π by $\frac{1}{n!}\pi_n$ constructs a morphism of mixed complexes from $HH(A/k)$ to $(\Omega^\bullet_{A/k}, 0, d)$, which is a quasi-isomorphism of underlying complexes. Therefore it induces quasi-isomorphisms on HC, HC^-, and HP by Remark 2.21, whence Example 2.20(ii) completes the proof. ☐

2.3.2 Group algebras

Let k be a commutative base ring and let G be an abelian group, written multiplicatively. Here we describe some relations between the homology of G and the cyclic homology of the group algebra $k[G]$.

The *cyclic bar construction*

$$B^{\mathrm{cycl}}G = \qquad G \leftleftarrows G^2 \leftleftarrows G^3 \leftleftarrows \cdots$$

is the cyclic set having face and degeneracy maps, and cyclic operator, as in the Hochschild complex, i.e., the maps $(g_0, \dots, g_n) \mapsto g_0 \otimes \cdots \otimes g_n$ define an inclusion of cyclic objects $B^{\mathrm{cycl}}G \subseteq HH(k[G]/k)$. This is clearly an equality after linearising $B^{\mathrm{cycl}}G$ to turn it into a cyclic k-module:

$$k[B^{\mathrm{cycl}}G] = HH(k[G]/k).$$

To analyse $B^{\mathrm{cycl}}G$ we decompose it as follows. Firstly, the simplicial set

$$BG = \quad 1 \leftleftarrows G \leftleftarrows G^2 \leftleftarrows \cdots$$

denotes the usual bar construction; it may be equipped with the structure of a cyclic set by declaring the cyclic operator $t_n \in \mathbb{Z}/(n+1)\mathbb{Z}$ to act on $B_nG = G^n$ as

$$(g_1, \ldots, g_n) \mapsto ((g_1 \cdots g_n)^{-1}, g_1, \ldots, g_{n-1}).$$

More generally, for any $z \in G$ there is a "twisted" bar construction $B(G, z)$ which is again a cyclic set: as a simplicial set it is the same as BG, but the cyclic operator t_n is now defined by $(g_1, \ldots, g_n) \mapsto (z(g_1 \cdots g_n)^{-1}, g_1, \ldots, g_{n-1})$; obviously $B(G, 1) = BG$.

For any $z \in G$ there is an inclusion of cyclic sets

$$B(G, z) \hookrightarrow B^{\mathrm{cylc}}G, \qquad (g_1, \ldots, g_n) \mapsto (z(g_1 \cdots g_n)^{-1}, g_1, \ldots, g_n)$$

with image $\{(g_0, \ldots, g_n) \in B_n^{\mathrm{cylc}}G : g_0 \cdots g_n = z)\}$; these assemble to define a decomposition of the cyclic bar construction $B^{\mathrm{cylc}}G = \bigsqcup_{z\in G} B(G, z)$. Forgetting the cyclic structure, this decomposition may be written as one of simplicial sets $B^{\mathrm{cylc}}G = \bigsqcup_{z\in G} zBG$, where zBG is the left translation of BG.

By then k-linearising we see that we have defined a direct sum decomposition of cyclic k-modules

$$HH(k[G]/k) = \bigoplus_{z\in G} k[B(G, z)].$$

Forgetting the cyclic structure, this may be written as a direct sum decomposition of simplicial modules $HH(k[G]/k) = \bigoplus_{z\in G} zk[BG] = k[G] \otimes_k k[BG]$, which we summarise in the next lemma:

Lemma 2.23. *The natural map of simplicial $k[G]$-modules*

$$k[G] \otimes_k k[BG] \longrightarrow HH(k[G]/k), \qquad f \otimes (g_1, \ldots, g_n) \mapsto f(g_1 \cdots g_n)^{-1} \otimes g_1 \otimes \cdots \otimes g_n,$$

where $g_1, \ldots, g_n \in G$ and $f \in k[G]$, is an isomorphism. Passing to homology yields a natural isomorphism of graded $k[G]$-algebras

$$k[G] \otimes_k H_*(G, k) \xrightarrow{\simeq} HH_*(k[G]/k).$$

Proof. The only statement still requiring justification is the homology calculation, which follows from the fact that the simplicial k-module $k[BG]$ models the group homology $R\Gamma(G, k)$, where G acts trivially on k. □

The cyclic structure on the twisted bar constructions is somewhat subtle, so we focus now only on the cyclic k-module $k[BG]$, whose associated cyclic homology etc. we denote by $HC(G/k)$ etc. It is known that the S^1-action on the geometric realisation of BG is trivial whence the cyclic homology and its variants behave as in Example 2.20 (since we do not need the following proposition, we refer to [29, §7.3.9 & Prop. 7.4.8] for further explanation):

Proposition 2.24. *There are natural equivalences*

$$HC^-(G/k) \simeq \prod_{i\le 0} R\Gamma(G, k)[2i], \qquad HP(G/k) \simeq \prod_{i\in\mathbb{Z}} R\Gamma(G, k)[2i], \qquad HC(A/k) \simeq \prod_{i\ge 0} R\Gamma(G, k)[2i].$$

2.4 The cotangent complex and Hochschild homology for non-flat algebras

We briefly summarise the theory of the cotangent complex; the classical standard references are Quillen [35, 34], Illusie [23, 24], and M. André [1]. Let k be a commutative base ring and A a commutative k-algebra. Then there exists a simplicial resolution $P_\bullet \xrightarrow{\sim} A$ of A by polynomial k-algebras (possibly in infinitely many variables); this resolution is unique up to homotopy, whence the same is true of the simplicial A-module

$$\mathbb{L}_{A/k} := \Omega^1_{P_\bullet/k} \otimes_{P_\bullet} A$$

which is known as the *cotangent complex* of $k \to A$. By a standard abuse, $\mathbb{L}_{A/k}$ is often identified with its associated complex via Dold–Kan in $D(A)$ (whence the terminology). Similarly, the wedge powers of the cotangent complex are $\mathbb{L}^i_{A/k} := \bigwedge^i_A \mathbb{L}_{A/k} = \Omega^i_{P_\bullet/k} \otimes_{P_\bullet} A$.

Remark 2.25 (Left Kan extension)**.** Particularly in §3.1 it will be helpful to adopt a more modern perspective on the above construction, namely that of left Kan extensions. Let $\mathcal{D}$ be any ∞-category which admits sifted colimits (e.g., $D(k)$ or Sp), and write k-algs$_\Sigma$ for the category of polynomial k-algebras in finitely many variables. One says that a functor $\mathcal{F} : k\text{-algs} \to \mathcal{D}$ is *left Kan extended* (from k-algs$_\Sigma$) if the canonical map $\mathcal{F}(A) \to \varinjlim_{P\to A} \mathcal{F}(P)$ is an equivalence for each k-algebra A, where the colimit in $\mathcal{D}$ is taken over all $P \in k\text{-algs}_\Sigma$ mapping to A.

Conversely, given a functor $\mathcal{G} : k\text{-algs}_\Sigma \to \mathcal{D}$, we may left Kan extend it to $\mathbb{L}\mathcal{G} : k\text{-algs} \to \mathcal{D}$ by using the above formula: $\mathcal{G}(A) := \varinjlim_{P\to A} \mathcal{F}(P)$. We thus obtain an equivalence between functors $k\text{-algs}_\Sigma \to \mathcal{D}$ and left Kan extended functors $k\text{-algs} \to \mathcal{D}$.

Returning to the more concrete point of view, the left Kan extension is given as follows: first extend $\mathcal{G}$ to all polynomial k-algebras (possibly in infinitely many variables) by taking filtered colimits, then extend it to all k-algebras by the rule

$$\mathbb{L}\mathcal{G}(A) := |\mathcal{G}(P_\bullet)| = \text{geometric realisation of the simplicial } \mathcal{D}\text{-object } \mathcal{G}(P_\bullet)$$

where $P_\bullet \to A$ is any simplicial resolution of A by polynomial k-algebras. For example, $\mathbb{L}^i_{-/k}$ is precisely the left Kan extension of $\Omega^i_{-/k} : k\text{-algs}_\Sigma \to D(k)$ and could more correctly be denoted by $\mathbb{L}\Omega^i_{-/k}$.

Example 2.26 (Low degrees)**.** (i) For any k-algebra A and $n \geq 0$, the augmentation $\mathbb{L}^n_{A/k} \to \Omega^n_{A/k}$ induces $\pi_0(\mathbb{L}^n_{A/k}) \xrightarrow{\sim} \Omega^n_{A/k}$. Indeed, it is clearly surjective since $P_0 \twoheadrightarrow A$ induces $\Omega^1_{P_0/k} \twoheadrightarrow \Omega^1_{A/k}$; but the differential $d : P_\bullet \to \Omega^1_{P_\bullet/k}$ induces a derivative $d : A \to \pi_0(\mathbb{L}_{A/k})$ which in turn induces a map $\Omega^1_{A/k} \to \pi_0(\mathbb{L}_{A/k})$ splitting the surjection.

(ii) If $A = k/I$ is a quotient of k, then $\pi_0(\mathbb{L}_{A/k}) = 0$ and $\pi_1(\mathbb{L}_{A/k}) = I/I^2$. The vanishing of π_0 follows from (i), so it remains to justify the description of π_1. Unfortunately there doesn't follow straight from the definitions: one either relates the cotangent complex to algebra extensions [23, Corol. II.1.2.8.1], or else reads it off Quillen's "fundamental spectral sequence" [34, Thm. 6.3].

Next we summarise the fundamental abstract properties of the cotangent complex:

Proposition 2.27. *Let k be a commutative ring and A a commutative k-algebra.*

(i) (Base change) For any base change $k \to k'$ (where k' is commutative) such that $\mathrm{Tor}_i^k(A, k') = 0$ for $i > 0$, then $\mathbb{L}_{A/k} \otimes^{\mathbb{L}}_k k' \xrightarrow{\sim} \mathbb{L}_{A'/k}$ where $A' := A \otimes_k k'$.

(ii) (Künneth) For any other commutative k-algebra B such that $\mathrm{Tor}_i^k(A, B) = 0$ for $i > 0$, then $\mathbb{L}_{A\otimes_k B/k} \simeq (\mathbb{L}_{A/k} \otimes^{\mathbb{L}}_k B) \oplus (\mathbb{L}_{B/k} \otimes^{\mathbb{L}}_k A)$.

(iii) (Transitivity sequence) Given a commutative A-algebra B, the resulting sequence

$$\mathbb{L}_{A/k} \otimes^{\mathbb{L}}_A B \longrightarrow \mathbb{L}_{B/k} \longrightarrow \mathbb{L}_{B/A}$$

is a fibre sequence.

(iv) (Localisation) If $S \subseteq k$ is a multiplicative system, then $\mathbb{L}_{S^{-1}k/k} \simeq 0$

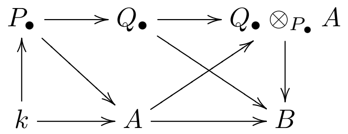
(v) (Étale) If A is étale over k, then $\mathbb{L}_{A/k} \simeq 0$.

(vi) (Smooth) If A is smooth over k then $\mathbb{L}_{A/k}$ is supported in degree zero, whence $\mathbb{L}_{A/k} \xrightarrow{\sim} \Omega^1_{A/k}[0]$.

Proof. For (i), just pick a simplicial resolution $P_\bullet \xrightarrow{\sim} A$ by polynomial k-algebras and observe that $P_\bullet \otimes_k k' \xrightarrow{\sim} A'$ is a resolution of A' by polynomial k'-algebras (indeed, the failure of it to be a resolution is precisely the higher Tors which we have assumed vanish); then use that $\Omega^1_{P_n/k} \otimes_k k' = \Omega^1_{P_n\otimes_k k'/k'}$ for each $n \geq 0$.

(ii): Let $P_\bullet \xrightarrow{\sim} A$ be as in the previous paragraph, and $Q_\bullet \xrightarrow{\sim} B$ similarly for B. Then $P_\bullet \otimes_k Q_\bullet$ is a simplicial resolution of $A \otimes_k B$ by polynomial k-algebras, whence $\mathbb{L}_{A\otimes_k B/k} = \Omega^1_{P_\bullet\otimes_k Q_\bullet} \otimes_{P_\bullet\otimes_k Q_\bullet} (A \otimes_k B)$. Rewriting this using $\Omega^1_{P_n\otimes_k Q_n/k} = \Omega^1_{P_n/k} \oplus \Omega^1_{Q_n/k}$ for each n easily gives the desired result.

To prove the transitivity sequence one first uses some standard properties of simplicial commutative rings (or, more precisely, of their model structure; we refer to the proof of [34, Thm. 5.1] for details) to build a commutative diagram



where $P_\bullet$ is a simplicial resolution of A by free k-algebras, $Q_\bullet$ is a simplicial resolution of B by a cofibrant $P_\bullet$-algebra (this means that $Q_\bullet$ is a retract of a free $P_\bullet$-algebra), and $Q_\bullet \otimes_{P_\bullet} A$ is a simplicial resolution of B by free A-algebras (or perhaps only retracts of free A-algebras). Since the maps $k \to P_n \to Q_n$ are all smooth, there is an exact Jacobi–Zariski sequence $0 \to \Omega^1_{P_n/k} \otimes_{P_n} Q_n \to \Omega^1_{Q_n/k} \to \Omega^1_{Q_n/P_n} \to 0$ for each n. Tensoring by $-\otimes_{Q_n} B$ and rearranging some of the tensor products gives a fibre sequence $\Omega^1_{P_\bullet/k} \otimes_{P_\bullet} A \to \Omega^1_{Q_\bullet/k} \otimes_{Q_\bullet} B \to \Omega^1_{Q_\bullet \otimes_{P_\bullet} A/A} \otimes_{Q_\bullet \otimes_{P_\bullet} A} B$, as desired.

To prove (iv) and (v) we follow a clever argument apparently due to M. André. Given a multiplicative system $S \subseteq k$, we have $S^{-1}k \otimes_k S^{-1}k = S^{-1}k$ and so $\mathbb{L}_{S^{-1}k/k} \otimes_k S^{-1}k = \mathbb{L}_{S^{-1}k/k} \otimes_{S^{-1}k} (S^{-1}k \otimes_k S^{-1}k) = \mathbb{L}_{S^{-1}k/k}$; but part (i) shows that $\mathbb{L}_{S^{-1}k/k} \otimes_k S^{-1}k = \mathbb{L}_{S^{-1}k \otimes_k S^{-1}k/S^{-1}k} = \mathbb{L}_{S^{-1}k/S^{-1}k} = 0$.

For (v), we recall that étale means that $k \to A$ is flat and that $\mu : A \otimes_k A \to A$ induces an open immersion on Spec. Therefore, for any prime ideal $\mathfrak{p} \subseteq A$ we have $A_\mathfrak{p} = (A \otimes_k A)_\mathfrak{q}$ with $\mathfrak{q} := \mu^{-1}(\mathfrak{p})$, and so

$$\begin{aligned}
\mathbb{L}_{A/k} \otimes_A A_\mathfrak{p} &= \mathbb{L}_{A/k} \otimes_A (A \otimes_k A) \otimes_{A \otimes_k A} (A \otimes_k A)_\mathfrak{q} \\
&= \mathbb{L}_{A \otimes_k A/A} \otimes_{A \otimes_k A} (A \otimes_k A)_\mathfrak{q} && \text{(by part (i))} \\
&= \mathbb{L}_{(A \otimes_k A)_\mathfrak{q}/A} && \text{(by transitivity and } \mathbb{L}_{(A \otimes_k A)_\mathfrak{q}/A \otimes_k A} = 0 \text{ by (iv))} \\
&= \mathbb{L}_{A_\mathfrak{p}/A} \\
&= 0 && \text{(by (iv))}
\end{aligned}$$

Since this holds for all prime ideals $\mathfrak{p} \subseteq A$, we have $\mathbb{L}_{A/k} = 0$.

To prove (vi) we then argue as follows: it is enough to prove the result Zariski locally on Spec A (using (i)), so we may suppose that A is étale over $k[t_1, \ldots, t_d]$ for some $d \geq 0$. But then the result follows at once from the following: $\mathbb{L}_{k[\underline{t}]/k} = \Omega^1_{k[\underline{t}]/k}$ (since $k[\underline{t}]$ serves as a polynomial resolution of itself); $\mathbb{L}_{A/k[\underline{t}]} = 0$ by (v); the transitivity sequence for $k \to k[\underline{t}] \to A$; the well-known fact that $\Omega_{k[\underline{t}]/k} \otimes_{k[\underline{t}]} A \xrightarrow{\simeq} \Omega^1_{A/k}$ since $k[\underline{t}] \to A$ is étale. ☐

Our interest in the the cotangent complex stems from the following relation to Hochschild homology

Proposition 2.28 (Hochschild–Kostant–Rosenberg filtration)**.** *Let A be a commutative k-algebra.*

(i) Then the Hochschild complex $HH(A/k)$ (viewed as an object of $D(A)$) admits a natural, complete, descending $\mathbb{N}$-indexed filtration whose i^{th} graded piece is equivalent to $\mathbb{L}^i_{A/k}[i]$, for $i \geq 0$.

(ii) Similarly, $HC(A/k)$ admits a natural, complete, descending $\mathbb{N}$-indexed filtration whose i^{th}-graded piece is $\bigoplus_{n\geq 0} \mathbb{L}^i_{A/k}[i+2n]$, for $i \geq 0$.

Remark 2.29 (Complete filtrations)**.** Although we have already used terminology surrounding filtrations in §2.2, we did not give definitions; we do so now. Let us begin with the case of an honest complex C over a ring; by a *descending, $\mathbb{N}$-indexed filtration* we mean a descending chain of subcomplexes $C = \mathrm{Fil}^0 C \supseteq \mathrm{Fil}^1 C \supseteq \cdots$. To say that it is *complete* means that the canonical map $C_n \to \varprojlim_i C_n/\mathrm{Fil}^i C_n$ is an isomorphism for each degree n.

However, even when C is an explicit complex like $HH(A/k)$, we always prefer to view C merely as an object of the derived category (where it does not make sense to discuss subcomplexes or the individual terms C_n); in that case a descending, $\mathbb{N}$-indexed filtration means simply that we are given complexes and maps between then $C = \mathrm{Fil}^0 C \leftarrow \mathrm{Fil}^1 C \leftarrow \mathrm{Fil}^2 C \leftarrow \cdots$, and complete means that the canonical map $C \to \mathrm{Rlim}_i C/\mathrm{Fil}^i C$ is an equivalence (informally, we are asking that C can be completely recovered from all the $C/\mathrm{Fil}^i C$). Here we write $C/\mathrm{Fil}^i C$ for the cofiber of the map $\mathrm{Fil}^i C \to C$, so the completeness condition is equivalent to asking that $\mathrm{Rlim}_i \mathrm{Fil}^i C \simeq 0$.

Occasionally we will also encounter ascending filtrations, in which case *exhaustive* means that the filtered colimit up the filtration coincides (up to equivalence) with the complex itself.

We have deliberately stated the proposition without assuming that A is necessarily flat over k; therefore we must first define Hochschild and cyclic homology in this greater degree of generality. Although the definitions of §2.1–2.2 do work without assuming such flatness, the resulting theory would less closely related to topological Hochschild homology (in which all tensor products will be automatically derived, thereby automatically overcoming the flatness issues) and Proposition 2.28 would not hold.

To define the Hochschild and cyclic homologies of a k-algebra which is not necessarily flat, we essentially replace $\otimes_k$ by $\otimes_k^{\mathbb{L}}$. More precisely, we define the Hochschild homology $HH(A/k)$ to be the diagonal of the bisimplicial k-module $HH(P_\bullet/k)$, where $P_\bullet \to A$ is a simplicial resolution of A by polynomial k-algebras. In other words (recalling that the diagonal of a bisimplicial object models the geometric realisation), we are defining $HH(-/k) : k\text{-algs} \to D(k)$ to be the left Kan extension of $HH(-/k) : k\text{-algs}_\Sigma \to D(k)$. Note that if A is flat over k, then the augmentations $P_\bullet^{\otimes_k n} \to A^{\otimes_k n}$ are equivalences for all $n \geq 0$, whence the diagonal of the bisimplicial k-module $HH(P_\bullet/k)$ is equivalent to $HH(A/k)$ as it was defined in the flat case, i.e., the two definitions agree in the flat case.

Similarly, we define $HC(A/k)$ as the totalisation of the simplicial cochain complex $HC(P_\bullet/k)$, or in other words as the left Kan extension of $HC(-/k) : k\text{-algs}_\Sigma \to D(k)$. This agrees with the old definition if A is flat over k, and the periodicity sequence and increasing filtration of Remark 2.13 formally remain valid for general A.

One can alternatively define $HH(A/k)$ and $HC(A/k)$ using a resolution of A by a flat differential graded k-algebra; this approach was adopted in [37, 38], for which reason $HH(A/k)$ of non-flat algebras is sometimes called Shukla homology.

Proof of Proposition 2.28. (i): Since $HH(A/k)$ is obtained by totalising the bisimplicial object $HH(P_\bullet/k)$, where $P_\bullet \to A$ is a simplicial resolution by free k-algebras, it follows formally that the Postnikov filtration on each $HH(P_n/k)$ naturally induces a complete, descending $\mathbb{N}$-indexed filtration on $HH(A/k)$ whose i^{th} graded piece is given by $HH_i(P_\bullet/k)[i]$, $i \geq 0$. But each P_n is a polynomial k-algebra, so Theorem 2.8 implies that $HH_i(P_n/k) \cong \Omega^i_{P_n/k}$; therefore $HH_i(P_\bullet/k) \cong \Omega^i_{P_\bullet/k}$, which we may rewrite up to equivalence as $\Omega^i_{P_\bullet/k} \otimes^{\mathbb{L}}_{P_\bullet} P_\bullet = \Omega^i_{P_\bullet/k} \otimes_{P_\bullet} A$ since each $\Omega^i_{P_n/k}$ is a free P_n-module; but the latter simplicial module is precisely $\mathbb{L}^i_{A/k}$.

In short, the desired filtration on $HH(-/k) : k\text{-algs} \to D(k)$ is the left Kan extension of the Postnikov filtration on $HH(-/k) : k\text{-algs}_\Sigma \to D(k)$.

(ii): We begin by supposing that $A \in k\text{-algs}_\Sigma$, or more generally that A is smooth over k. In terms of $HC(A/k)$ as the totalisation of the BC-bicomplex, we define $\mathrm{Fil}^i_{\mathrm{HKR}} HC(A/k)$ to be its subcomplex obtained by totalising

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
A^{\otimes_k i+3} & \xleftarrow{B} & A^{\otimes_k i+2} & \xleftarrow{B} & \operatorname{Ker} b & \xleftarrow{B} & 0 \\
\downarrow b & & \downarrow b & & \downarrow b & & \\
A^{\otimes_k i+2} & \xleftarrow{B} & \operatorname{Ker} b & \xleftarrow{B} & 0 & & \\
\downarrow b & & \downarrow b & & & & \\
\operatorname{Ker} b & \xleftarrow{B} & 0 & & & & \\
\downarrow b & & & & & & \\
0 & & & & & &
\end{array}$$

(i.e., canonical truncation on the columns). The i^{th} graded step of this filtration is clearly the totalisation of

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & 0 & \xleftarrow{B} & A^{\otimes_k i+2}/\operatorname{Ker} b & \xleftarrow{B} & \operatorname{Ker} b & \xleftarrow{B} & 0 \\
 & & \downarrow b & & \downarrow b & & \downarrow b \\
 A^{\otimes_k i+2}/\operatorname{Ker} b & \xleftarrow{B} & \operatorname{Ker} b & \xleftarrow{B} & 0 \\
 \downarrow b & & \downarrow b \\
 \operatorname{Ker} b & \xleftarrow{B} & 0 \\
 \downarrow b \\
 0
\end{array}
$$

i.e., $\bigoplus_{n\geq 0}[\cdots \leftarrow 0 \leftarrow \underline{\operatorname{Ker} b} \leftarrow A^{\otimes_k i+2}/\operatorname{Ker} b \leftarrow 0 \leftarrow \cdots][i+2n]$ where we underline the term in degree 0. But each of the complexes in the direct sum is equivalent to $HH_i(A/k) \cong \Omega^i_{A/k}$, whence the graded piece is equivalent to $\bigoplus_{n\geq 0} \Omega^i_{A/k}[i+2n]$ as desired.

To construct the analogous filtration on $HH(A/k)$, with graded pieces $\bigoplus_{n\geq 0} \mathbb{L}^i_{A/k}[i+2n]$, for arbitrary A there are two (essentially equivalent) ways to argue: either rewrite the above argument replacing all the columns by $HH(P_\bullet/k)$, or else just formally declare the desired filtration on $HC(-/k) : k\text{-algs} \to D(k)$ to be the left Kan extension of the just-construction filtration on $HC(-/k) : k\text{-algs}_\Sigma \to D(k)$.

It remains to check that the filtration on $HC(A/k)$ is complete, i.e., that $\operatorname{Rlim}_i \operatorname{Fil}^i_{\mathrm{HKR}} HC(A/k) \simeq 0$. But this is a formal consequence of the fact that $\operatorname{Fil}^i_{\mathrm{HKR}} HC(A/k)$ is supported in homological degrees $\geq i$ by construction. ☐

Remark 2.30. The antisymmetrisation maps $\Omega^n_{A/k} \to HH_n(A/k)$ from Theorem 2.8 continue to exist even if A is not flat over k; indeed, they are precisely the edge maps in the spectral sequence which arises from the filtration of Proposition 2.28.

Remark 2.31 (Transitivity in general)**.** Let $A \to B$ be a morphism of k-algebras, where A is commutative. Then there is a natural equivalence of simplicial A-algebras

$$HH(B/k) \otimes^{\mathbb{L}}_{HH(A/k)} A \xrightarrow{\sim} HH(B/A),$$

generalising the isomorphism of Remark 2.7 in the flat case. This can be proved either by left Kan extending the aforementioned isomorphism, or else picking compatible resolutions and arguing as in the proof of Proposition 2.27(iii).

Example 2.32. The Hochschild homology groups of the $\mathbb{Z}$-algebra $\mathbb{F}_p$ are given by

$$HH_n(\mathbb{F}_p/\mathbb{Z}) \cong \begin{cases} \mathbb{F}_p & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Regarding multiplicative structure, $HH_*(\mathbb{F}_p/\mathbb{Z}) = \mathbb{F}_p\langle u\rangle$ (divided power algebra on a single variable u) with $u \in HH_2(\mathbb{F}_p/\mathbb{Z})$ any basis element. In particular, the powers $u^2, \ldots, u^{p-1}$ serve as basis elements for $HH_4(\mathbb{F}_p/\mathbb{Z}), \ldots, HH_{2(p-1)}(\mathbb{F}_p/\mathbb{Z})$ respectively, but $u^p = 0$ in $HH_{2p}(\mathbb{F}_p/\mathbb{Z})$. This is usually viewed as pathological behaviour, which will be fixed by topological Hochschild homology.

There are two ways to prove this. Either use the HKR filtration of Theorem 2.28 and a standard calculation of the wedge powers of the cotangent complex for a ring modulo a regular element (see Remark 3.9, which also shows that $HH_2(\mathbb{F}_p/\mathbb{Z})$ naturally identifies with $\pi_1(\mathbb{L}_{\mathbb{F}_p/\mathbb{Z}}) = p\mathbb{Z}/p^2\mathbb{Z}$; this gives a preferred choice of u, namely the class of $p \bmod p^2$), or use the standard resolution of $\mathbb{F}_p$ by the Koszul complex $K(p;\mathbb{Z}) = [\mathbb{Z} \xleftarrow{p} \mathbb{Z}]$ (which is a flat cdg $\mathbb{Z}$-algebra) and compute $HH(\mathbb{F}_p/\mathbb{Z})$ from the point of view of Shukla homology. We refer to [28, Prop. 2.6].

3 Hochschild and cyclic homology of $\mathbb{F}_p$-algebras

In Theorem 2.22 we saw that the cyclic homology theories of a smooth algebra in characteristic zero were given by de Rham cohomology and its Hodge filtration. The goal of this section is twofold; firstly, we will present a proof of the following analogous result in characteristic p:

Theorem 3.1. *Let k be a perfect field of characteristic p and R a smooth k-algebra. Then $HC^-(R/k)$, $HP(R/k)$, and $HC(R/k)$ admit natural complete, descending, $\mathbb{Z}$-indexed filtrations whose i^{th} graded pieces are respectively given by*

$$\Omega^{\geq i}_{R/k}[2i], \qquad \Omega^{\bullet}_{R/k}[2i], \qquad \Omega^{\leq i}_{R/k}[2i].$$

In short, Theorem 2.22 remains true in characteristic p except that the filtration need not be split. Theorem 3.1 is far from the best possible generalisation of Theorem 2.22 beyond characteristic zero; in fact, Antieau has generalised it to smooth algebras over arbitrary base rings [2] (and has even removed the smoothness hypothesis at the expense of replacing de Rham cohomology by its derived version). However, the second goal of this section is to introduce the reader to the methods of [8], in which "motivic" filtrations are constructed on the cyclic homology and its topological variants of $\mathbb{F}_p$- and p-adic algebras. Therefore we will prove Theorem 3.1 following the main technique of [8], namely descent to the case of quasiregular semiperfect rings.

In §3.1 we present this technique by explaining how suitable cohomology theories can be determined by their behaviour on certain semiperfect rings. This technique is then applied in §3.2 in order to prove Theorem 2.22. Finally in §3.4 the methods are abstracted and extended by introduction of the quasisyntomic site.

3.1 Flat descent via quasiregular semiperfect algebras

Given a commutative base ring k, suppose that we have a functor

$$\mathcal{F} : k\text{-algs} \to D(\mathbb{Z}) \text{ or } D(k) \text{ or } \mathrm{Sp} \text{ or in general any } \infty\text{-category } \mathcal{D} \text{ with sifted colimits.}$$

In Remark 2.25 we recalled the theory of left Kan extension, which allows us to make precise the idea that $\mathcal{F}$ might be determined by its value on polynomial algebras, and so in particular by its value on smooth algebras. Now we consider a converse problem: can we recover the behaviour of $\mathcal{F}$ on smooth algebras from its values on some class of "large algebras" (where, perhaps perversely, we hope that $\mathcal{F}$ is easier to understand)?

Definition 3.2. $\mathcal{F}$ is said to satisfy *flat descent* (or to be an *fpqc ∞-sheaf,* where *fpqc* means fidèlement plat et quasi-compact) if, for every faithfully flat map $A \to B$ of k-algebras, the induced morphism

$$\mathcal{F}(A) \to \lim \left(\mathcal{F}(B) \rightrightarrows \mathcal{F}(B \otimes_A B) \substack{\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow} \mathcal{F}(B \otimes_A B \otimes_A B) \substack{\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow} \cdots \right)$$

in $\mathcal{D}$ is an equivalence. Inside the bracket we are applying $\mathcal{F}$ termwise to the usual Cech nerve

$$\mathrm{Cech}(B/A) := \qquad B \rightrightarrows B \otimes_A B \substack{\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow} B \otimes_A B \otimes_A B \substack{\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow} \cdots$$

(which is a cosimplicial object in $\mathcal{D}$), and the recall that the limit may be interpreted as totalisation in case $\mathcal{D} = D(\mathbb{Z})$ or $D(k)$.

Now we suppose that k is a perfect field of characteristic p and explain how flat descent allows us to answer the converse problem we have posed. Let R be a smooth k-algebra, and recall first that the Frobenius morphism $\varphi : R \to R$ is therefore flat. Indeed, to prove this we may work locally on $\operatorname{Spec} R$ and so assume that there exists an étale morphism $k[t_1, \dots, t_d] \to R$ for some $d \geq 0$; the étaleness implies that the diagram

$$\begin{array}{ccc} k[t_1, \dots, t_d] & \longrightarrow & R \\ \varphi \big\uparrow & & \big\uparrow \varphi \\ k[t_1, \dots, t_d] & \longrightarrow & R \end{array}$$

is a pushout, and so the desired flatness follows from the obvious flatness of φ on $k[t_1, \dots, t_d]$. By taking the colimit over iterations of φ, we see that the colimit perfection $R_{\mathrm{perf}} := \varinjlim_{\varphi} R$ is a flat R-algebra. Moreover φ always induces a homeomorphism on Spec, so $\operatorname{Spec} R_{\mathrm{perf}} \to \operatorname{Spec} R$ is a homeomorphism, in particular surjective; so we have shown that $R \to R_{\mathrm{perf}}$ is faithfully flat.

Assuming that $\mathcal{F}$ satisfies flat descent, we deduce that

$$\mathcal{F}(R) \xrightarrow{\sim} \lim \big(\mathcal{F}(R_{\mathrm{perf}}) \rightrightarrows \mathcal{F}(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{F}(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots \big),$$

which describes $\mathcal{F}(R)$ in terms of $\mathcal{F}(R_{\mathrm{perf}})$ and all the $\mathcal{F}(R_{\mathrm{perf}} \otimes_R \cdots \otimes_R R_{\mathrm{perf}})$. The tensor powers $R_{\mathrm{perf}} \otimes_R \cdots \otimes_R R_{\mathrm{perf}}$ are no longer perfect, but they are locally quotients of perfect k-algebras by regular sequences. Indeed, assuming as in the previous paragraph that there is an étale morphism $k[t_1, \dots, t_d] \to R$, one obtains

$$R_{\mathrm{perf}} \otimes_R \cdots \otimes_R R_{\mathrm{perf}} = \big(k[\underline{t}]_{\mathrm{perf}} \otimes_{k[\underline{t}]} \cdots \otimes_{k[\underline{t}]} k[\underline{t}]_{\mathrm{perf}} \big) \otimes_{k[\underline{t}]} R,$$

where $k[\underline{t}]_{\mathrm{perf}} = k[t_1^{1/p^\infty}, \dots, t_d^{1/p^\infty}]$. But the large bracketed term is indeed a perfect k-algebra modulo a regular sequence, for example

$$k[t^{1/p^\infty}] \otimes_{k[t]} k[t^{1/p^\infty}] \cong k[x^{1/p^\infty}, y^{1/p^\infty}]/(x-y), \qquad t^{1/p^j} \otimes 1 \mapsto x^{1/p^j}, \quad 1 \otimes t^{1/p^j} \mapsto y^{1/p^j}.$$

For practical reasons, being locally a quotient of a perfect algebra modulo a regular sequence is not a good class of rings, since regular sequences are not the right notion for non-Noetherian rings; it is better to work instead with the following wider class, which avoids any finiteness hypotheses:

Definition 3.3. Quillen calls an ideal I of a ring A *quasiregular* if and only if I/I^2 is a flat A/I-module and $\pi_n(\mathbb{L}_{A/I\,/A}) = 0$ for $n > 1$ (whence $\mathbb{L}_{A/I\,/A} \simeq I/I^2[1]$ by Example 2.26) [34, Thm. 6.13]. For example, if I is locally generated by a regular sequence then it is quasiregular.

We say that an $\mathbb{F}_p$-algebra A is *quasiregular semiperfect* (qrsp) if and only if it is semiperfect (i.e., the Frobenius $\varphi : A \to A$ is surjective) and $\mathbb{L}_{A/\mathbb{F}_p}$ is a flat A-module supported in homological degree 1.

Lemma 3.4. *An $\mathbb{F}_p$-algebra A is quasiregular semiperfect if and only if there exists a perfect $\mathbb{F}_p$-algebra S and a quasiregular ideal $I \subseteq S$ such that $S/I = A$.*

Proof. We first recall the following general result: if $S' \to S$ is any morphism between perfect $\mathbb{F}_p$-algebras, then $\mathbb{L}_{S/S'} \simeq 0$. Indeed, let $P_\bullet \to S$ be a simplicial resolution by polynomial S'-algebras and observe that the absolute Frobenius φ induces the zero map on $\Omega^1_{P_\bullet/S'}$, since $\varphi(df) = df^p = pf^{p-1}dp = 0$; but S and S' are perfect, so this zero map $\varphi : \mathbb{L}_{S/S'} \to \mathbb{L}_{S/S'}$ is also an equivalence.

Both hypotheses in the statement of the lemma include that A is semiperfect, so let S be any perfect $\mathbb{F}_p$-algebra surjecting onto A, and let I denote the kernel; the standard choice is $A^\flat := \varprojlim_{\varphi} A$, the inverse limit perfection (aka. tilt) of A. Then $\mathbb{L}_{S/\mathbb{F}_p} \simeq 0$ by the previous paragraph, so the transitivity sequence shows that $\mathbb{L}_{A/\mathbb{F}_p} = \mathbb{L}_{A/S}$; by definition this is a flat A-module supported in homological degree 1 if and only if I is quasiregular. ☐

In light of the previous lemma and definition, the prior discussion may therefore be summarised by the following proposition:

Proposition 3.5. *Let k be a perfect field of characteristic p and $\mathcal{F} : k\text{-algs} \to \mathcal{D}$ (where $\mathcal{D} = D(\mathbb{Z})$ or $D(k)$ or Sp) a functor satisfying flat descent. Then, for any smooth k-algebra R, we have*

$$\mathcal{F}(R) \xrightarrow{\sim} \lim \big(\mathcal{F}(R_{\mathrm{perf}}) \rightrightarrows \mathcal{F}(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{F}(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots \big),$$

and each term $R_{\mathrm{perf}} \otimes_R \cdots \otimes_R R_{\mathrm{perf}}$ appearing in the Cech nerve is quasiregular semiperfect.

And then here is the resulting filtration on $\mathcal{F}(R)$ which we have promised to construct in Theorem 3.1 (we remark that in all the cases of interest, $\mathcal{F}_i$ will vanish on any qrsp algebra for i odd and so we will discard the odd terms and reindex the filtration):

Corollary 3.6. *Under the same hypotheses as Proposition 3.5, suppose that $\mathcal{D} = D(\mathbb{Z})$, $D(k)$, or* Sp. *Then $\mathcal{F}(R)$ has a natural, complete, descending $\mathbb{Z}$-indexed filtration with i^{th}-graded piece given by the $[i]$-shift of the cosimplicial abelian group*

$$\mathcal{F}_i(R_{\text{perf}}) \rightrightarrows \mathcal{F}_i(R_{\text{perf}} \otimes_R R_{\text{perf}}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathcal{F}_i(R_{\text{perf}} \otimes_R R_{\text{perf}} \otimes_R R_{\text{perf}}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots$$

(or more precisely by its totalisation in $\mathcal{D}$, i.e., associated cochain complex via Dold–Kan), where $\mathcal{F}_i(-) := \pi_i \mathcal{F}(-)$.

Proof. For each $A = R_{\text{perf}} \otimes_R \cdots \otimes_R R_{\text{perf}}$, we equip $\mathcal{F}(A)$ with its natural complete, descending, $\mathbb{Z}$-indexed Postnikov/Whitehead filtration $\tau_{>i}\mathcal{F}(A)$. This formally induces a filtration on $\mathcal{F}(R)$ by Proposition 3.5, which is still complete since the limits $\mathcal{F}(A) \xrightarrow{\sim} \lim_i \tau_{\leq i}\mathcal{F}(A)$ may be commuted through the totalisation. The graded pieces of this filtration on $\mathcal{F}(R)$ are precisely

$$\lim \big(\mathcal{F}_i(R_{\text{perf}}) \rightrightarrows \mathcal{F}_i(R_{\text{perf}} \otimes_R R_{\text{perf}}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathcal{F}_i(R_{\text{perf}} \otimes_R R_{\text{perf}} \otimes_R R_{\text{perf}}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots\big)[i],$$

as desired. □

3.2 Cyclic homology of quasiregular semiperfect algebras

We have seen in §3.1, especially Proposition 3.5 and Corollary 3.6, how flat descent provides a general method for constructing filtrations, by descending the Postnikov filtration from quasiregular semiperfect $\mathbb{F}_p$-algebras. Now we wish to put this technique into practice to prove Theorem 3.1. The first step is to check that cyclic homology and its variants satisfy the necessary axiom for the technique, namely:

Lemma 3.7. *For any commutative ring k, the $D(k)$-valued functors*

$$\mathbb{L}^i_{-/k}, \quad HH(-/k), \quad HC(-/k), \quad HC^-(-/k), \quad HP(-/k)$$

on k-algs *satisfy flat descent.*

Proof. The fact that $\mathbb{L}^i_{-/k}$ satisfies flat descent is due to Bhatt; his first proof is given in [4], and his shorter proof may be found in [8, Thm. 3.1].

Since the property of satisfying flat descent is itself a derived limit and hence closed under derived limits (including fibre sequences), Proposition 2.28 and the previous paragraph imply that $HH(-/k)$ and $HC(-/k)$ satisfy flat descent. The complete filtration on $HC^-(-/k)$ from Remark 2.14 then similarly implies that $HC^-(-/k)$ satisfies flat descent, then deduce it for $HP(-/k)$ thanks to the norm fibre sequence of Remark 2.15. □

The second step is ideally to compute the cyclic homology and its variants of quasiregular semiperfect algebras; we begin with the following coarse information:

Lemma 3.8. *Let A be a qrsp $\mathbb{F}_p$-algebra. Then $HH_*(A/\mathbb{F}_p)$, $HC^-_*(A/\mathbb{F}_p)$, $HP_*(A/\mathbb{F}_p)$, and $HC_*(A/\mathbb{F}_p)$ are all supported in even degrees. Moreover, the $\mathbb{F}_p$-algebra $HP_0(A/\mathbb{F}_p)$ admits a complete, descending, $\mathbb{N}$-indexed filtration by ideals such that*

$$\operatorname{Fil}^i \cong HC^-_{2i}(A/\mathbb{F}_p), \qquad HP_0(A/\mathbb{F}_p)/\operatorname{Fil}^i \cong HC_{2i-2}(A/\mathbb{F}_p), \qquad \operatorname{gr}^i \cong HH_{2i}(A/\mathbb{F}_p) \cong \pi_i(\mathbb{L}^i_{A/\mathbb{F}_p})$$

for all $i \geq 0$.

Remark 3.9. In the setting of the lemma, the hypothesis is that $\mathbb{L}_{A/\mathbb{F}_p} = N[1]$ is given by a flat A-module supported in homological degree 1. Standard results on divided wedge powers then imply that $\mathbb{L}^i_{A/\mathbb{F}_p}$ is given by a flat A-module supported in homological degree i, namely $\Gamma^i_A(N)$, compatibly with multiplication as i varies [23, Prop. I.4.3.2.1].

Here $\Gamma^*_A(N) = \bigoplus_{i\geq 0} \Gamma^i_A(N)$ is the divided power algebra associated to the module N. Since N is flat, each A-module $\Gamma^i_A(N)$ is isomorphic to $\operatorname{Sym}^i_A(N)$, but the algebra structure on $\Gamma^*_A(N)$ is given by

$$x \cdot y := \tfrac{(i+j)!}{i!j!} xy \qquad x \in \Gamma^i_A(N),\, y \in \Gamma^j_A(N),$$

where $x \cdot y$ represents the product in $\Gamma^*_A(N)$ and xy represents the product in $\operatorname{Sym}^*_A(N)$.

Proof of 3.8. As mentioned in the remark, the fact that $\mathbb{L}^1_{A/\mathbb{F}_p} = N[1]$ implies that $\mathbb{L}^i_{A/\mathbb{F}_p} = \Gamma^i_A(N)[i]$. Therefore the spectral sequence associated to the filtration in Proposition 2.28 degenerates, implying that $HH_*(A/\mathbb{F}_p)$ is supported in even degrees and given as an A-algebra by $HH_{2*}(A/\mathbb{F}_p) = \Gamma^*_A(N)$.

The rest of the proof is quite general, we use only that $k := \mathbb{F}_p \to A$ is a morphism whose HH is supposed in even degrees. It follows formally that the various periodicity sequences in Remark 2.15 all break into short exact sequence, and that $HC_*(A/k)$, $HC^-_*(A/k)$, and $HP_*(A/k)$ are also supported in even degree:

$$0 \longrightarrow HH_{2i}(A/k) \xrightarrow{I} HC_{2i}(A/k) \xrightarrow{S} HC_{2i-2}(A/k) \longrightarrow 0$$

$$0 \longrightarrow HC^-_{2i+2}(A/k) \xrightarrow{S} HC^-_{2i}(A/k) \xrightarrow{h} HH_{2i}(A/k) \longrightarrow 0$$

$$0 \longrightarrow HC^-_{2i}(A/k) \xrightarrow{I} HP_{2i}(A/k) \xrightarrow{Sh} HC_{2i-2}(A/k) \longrightarrow 0$$

$$HP_{2i}(A/k) = \varprojlim_{s \text{ wrt } S} HC_{2i+2s}(A/k), \qquad S : HP_{2i}(A/k) \xrightarrow{\simeq} HP_{2i-2}(A/k)$$

There is therefore an identification of graded rings $HP_*(A/k) = HP_0(A/k)[u^{\pm 1}]$, with $u := S(1) \in HP_2(k/k)$ corresponding to the periodicity operator S. Furthermore, $I : HC^-_*(A/k) \to HP_*(A/k)$ is an inclusion of graded rings which is an isomorphism in degrees ≤ 0; therefore

$$\operatorname{Fil}^i HP_0(A/k) := u^{-i} I(HC^-_{2i}(A/k)) = S^n I((HC^-_{2i}(A/k)),$$

for $i \geq 0$, defines a complete decreasing filtration of $HP_0(A/k)$ by ideals with quotients and graded pieces given respectively by

$$HP_0(A/k)/\operatorname{Fil}^i \overset{h}{\xrightarrow{\simeq}} HC_{2i-2}(A/k), \qquad \operatorname{gr}^i \overset{h}{\xrightarrow{\simeq}} HH_{2i}(A/k).$$

□

Remark 3.10. The moral of the previous lemma and its proof is the following: since $HH_*(A/\mathbb{F}_p)$ is supported in even degree, the data of HC^-, HP, and HC are captured (up to extension problems) by the $\mathbb{F}_p$-algebra $HP_0(A/\mathbb{F}_p)$ and its filtration coming from HC^-.

The coarse information afforded by Lemma 3.8 will actually be sufficient to prove Theorem 3.1, but conceptually it is better to first try to explicitly identify the $\mathbb{F}_p$-algebra $HP_0(A/\mathbb{F}_p)$, for any qrsp algebra A. We now construct two possible candidates; they will both be equipped with a complete, descending, $\mathbb{N}$-indexed filtration by ideals.

Definition 3.11 (de Rham construction)**.** The first construction comes from the theory of derived de Rham cohomology. For each $i \geq 0$, we let $\mathbb{L}\Omega^{\leq i}_{-/\mathbb{F}_p} : \mathbb{F}_p\text{-algs} \to D(\mathbb{F}_p)$ be the left Kan extension of the naively truncated de Rham complex $\Omega^{\leq i}_{-/\mathbb{F}_p}$; recall from Remark 2.25 that this concretely means the following: given an $\mathbb{F}_p$-algebra A, we pick a simplicial resolution $P_\bullet \xrightarrow{\simeq} A$ by polynomial algebras over $\mathbb{F}_p$ (possibly in infinitely many variables), and define $\mathbb{L}\Omega^{\leq i}_{A/\mathbb{F}_p}$ to be the geometric realisation of the simplicial cochain complex $\Omega^{\leq i}_{P_\bullet/\mathbb{F}_p}$. The *Hodge completed derived de Rham complex* of the $\mathbb{F}_p$-algebra A is then defined to be

$$\widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p} := \operatorname{Rlim}_i \mathbb{L}\Omega^{\leq i}_{A/\mathbb{F}_p},$$

which is an algebra object in $D(\mathbb{F}_p)$ (possibly unbounded in both directions).

From the fibre sequences $\mathbb{L}^i_{A/\mathbb{F}_p} \to \mathbb{L}\Omega^{\leq i}_{A/\mathbb{F}_p} \to \mathbb{L}\Omega^{\leq i-1}_{A/\mathbb{F}_p}$ we see using Remark 3.9 that, assuming A is quasiregular semiperfect, then each $\mathbb{L}\Omega^{\leq i}_{A/\mathbb{F}_p}$ is supported in degree 0 and the transition maps are surjective as i increases; therefore $\widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p}$ is still supported in degree 0, given by an $\mathbb{F}_p$ algebra which is complete with respect to its induced filtration by the ideals $\operatorname{Ker}(\widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p} \twoheadrightarrow \mathbb{L}\Omega^{\leq i}_{A/\mathbb{F}_p})$ for $i \geq 0$.

Definition 3.12 (Divided power construction)**.** The second construction comes from the theory of divided powers. Assuming that A is quasiregular semiperfect (though in fact the definition only really requires that A be semiperfect), let $A^\flat := \lim_\varphi A$ be its inverse limit perfection and $I := \mathrm{Ker}(A^\flat \twoheadrightarrow A)$, so that $\mathbb{L}_{A/\mathbb{F}_p} = I/I^2[1]$ (see the proof of Lemma 3.4). Let $D_{A^\flat}(I)$ denote the divided power envelope of $A^\flat$ along I, and $\widehat{D}_S(I)$ its completion with respect to the divided power filtration (equipped with the induced filtration).

The following is the promised improvement of Lemma 3.8 which completely describes the cyclic, negative cyclic, and periodic cyclic homology groups of quasiregular semiperfect $\mathbb{F}_p$-algebras:

Proposition 3.13. *Let A be a quasiregular semiperfect $\mathbb{F}_p$-algebra. Then there are natural isomorphisms of filtered rings*

$$HP_0(A/\mathbb{F}_p) \cong \widehat{D}_{A^\flat}(I) \cong \widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p},$$

where $HP_0(A/\mathbb{F}_p)$ is equipped with the filtration coming from HC^- defined in Lemma 3.8.

Comments on the proof. We begin with a reality check that the isomorphisms are plausible, or more precisely valid for the associated graded rings. By construction of the filtered ring $\widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p}$, its graded pieces are precisely $\pi_i(\mathbb{L}^i_{A/\mathbb{F}_p})$ for $i \geq 0$. The same is true of the filtered ring $HP_0(A/\mathbb{F}_p)$, by Lemma 3.8. Meanwhile, it is a classical general result in the theory of divided powers that if I is generated by a regular sequence of $A^\flat$, then the filtered ring $D_{A^\flat}(I)$ has associated graded ring $\Gamma^*_{A^\flat}(N)$, where $N = I/I^2$. Since we know from Remark 3.9 that $\Gamma^*_{A^\flat}(N)$ and $\pi_*(\mathbb{L}^*_{A/\mathbb{F}_p})$ are naturally isomorphic, we have indeed shown that the three rings of the proposition have isomorphic graded rings (at least if I is generated by a regular sequence).

The obvious way to proceed is therefore to construct natural morphisms between the three rings of the proposition respecting the filtrations and to check that they induce the aforementioned isomorphisms on each graded piece; since the three rings are complete with respect to their filtrations, these morphisms will therefore be the desired isomorphisms. Fortunately the ring $D_{A^\flat}(I)$ has a universal property since it is a divided power envelope, so a priori we can construct maps out of it by showing that the target ring has suitable divided powers. Unfortunately it seems hard to directly write down natural divided power structures on $HP_0(A/\mathbb{F}_p)$ and $\widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p}$. See project F if you want to try.

Instead, Bhatt has constructed a comparison map $\widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p} \to \widehat{D}_{A^\flat}(I)$, by left Kan extending Berthelot's comparison from de Rham to crystalline cohomology, and shown that it is an isomorphism [5, Prop. 3.25 & Corol. 3.40] (assuming I is generated by a regular sequence). Some tricks in the style of the proofs of [8, Props. 8.12 & 8.15] should extend this to arbitrary quasiregular I.

To then compare to $HP_0(A/\mathbb{F}_p)$ there are two options. The first of these is based on Theorem 3.1, which we will prove in a moment without using the current proposition. Upon left Kan extending that theorem to all $\mathbb{F}_p$-algebras, it yields in particular natural isomorphisms $HC_{2i}(A/\mathbb{F}_p) \cong \mathbb{L}\Omega^{\leq i}_{A/\mathbb{F}_p}$; letting $i \to \infty$ gives $HP_0(A/\mathbb{F}_p) \cong \widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p}$, as desired.

The second approach for comparing to $HP_0(A/\mathbb{F}_p)$ is to use topological cyclic homology; indeed, we will see in Corollary 5.5 that the topological periodic cyclic homology $TP_0(A)$ is a p-adically complete, p-torsion-free ring such that $TP_0(A)/p = HP_0(A/\mathbb{F}_p)$. The p-adic completeness and p-torsion-freeness mean that existence of divided powers on $TP_0(A)$ (which would then induce divided powers on the quotient $HP_0(A/\mathbb{F}_p)$) becomes a condition on certain elements of the ring, rather than extra data to be specified. This allows the approach proposed in the second paragraph to be carried out; we refer to [8, Thm. 8.15] for the details. □

3.3 Cyclic homology of smooth algebras; proof of Theorem 3.1

This subsection is devoted to indicating the main ideas of proof of Theorem 3.1, so we let R denote a smooth algebra over a perfect field k of characteristic p. We will restrict to the central case of periodic cyclic homology; the analogous results for HC^- and HC follow by working with a suitable step of the filtrations instead, or respectively the quotient by a step of the filtration.

First proof of Theorem 3.1. If we allow ourselves to use Proposition 3.13, then Theorem 3.1 follows formally from the descent machinery as follows. Thanks to the general technique of Corollary 3.6, the verification in Lemma 3.7 that $HP(-/k)$ satisfies flat descent, and the fact from Lemma 3.8 that $HP_*(-/k)$ vanishes in odd degrees on any qrsp ring, we see that $HP(R/k)$ has a complete, descending, $\mathbb{Z}$-indexed filtration with graded pieces given by the $[2i]$-shifts of the cosimplicial k-module $HP_{2i}(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k)$

$$HP_{2i}(R_{\mathrm{perf}}/k) \rightrightarrows HP_{2i}(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}/k) \rightrightarrows HP_{2i}(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}/k) \rightrightarrows \cdots$$

We wish to show that this (or rather its totalisation) is equivalent to the de Rham complex $\Omega^\bullet_{R/k}$; we may suppose that $i = 0$, since HP_* is 2-periodic.

Proposition 3.13 shows that $HP_0(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k)$ is the same as

$$\widehat{\mathbb{L}\Omega}_{R_{\mathrm{perf}}/k} \rightrightarrows \widehat{\mathbb{L}\Omega}_{R_{\mathrm{perf}}\otimes_R R_{\mathrm{perf}}/k} \rightrightarrows \widehat{\mathbb{L}\Omega}_{R_{\mathrm{perf}}\otimes_R R_{\mathrm{perf}}\otimes_R R_{\mathrm{perf}}/k} \rightrightarrows \cdots.$$

But the totalisation of this is equivalent to $\widehat{\mathbb{L}\Omega}_{R/k} = \mathrm{Rlim}_i\, \mathbb{L}\Omega^{\le i}_{R/k}$ since the functor $\widehat{\mathbb{L}\Omega}_{-/k}$ satisfies flat descent (indeed, it is complete with respect to a filtration whose graded pieces are $\mathbb{L}^i_{-/k}$, and these satisfy flat descent by Lemma 3.7. But R is smooth over k, so the adjunction $\mathbb{L}\Omega^{\le i}_{R/k} \to \Omega^{\le i}_{R/k}$ is an equivalence and therefore $\widehat{\mathbb{L}\Omega}_{R/k} \simeq \mathrm{Rlim}\,\Omega^{\le i}_{R/k} = \Omega^\bullet_{R/k}$, as desired. ☐

Second proof of Theorem 3.1. We now sketch a proof of Theorem 3.1 which uses only the coarse information of Lemma 3.8 and does not rely on Proposition 3.13; as we explained in the proof of the latter, it can then be deduced via left Kan extension from the proof of the current theorem.

As in the first proof, the goal is to show that the totalisation of $HP_0(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k)$ is equivalent to $\Omega^\bullet_{R/k}$. The filtration on HP_0 of any qrsp ring from Lemma 3.8 induces a complete, descending, $\mathbb{N}$-indexed filtration on $HP_0(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k)$ with i^{th}-graded piece

$$\pi_i\mathbb{L}^i_{R_{\mathrm{perf}}/k} \rightrightarrows \pi_i\mathbb{L}^i_{R_{\mathrm{perf}}\otimes_R R_{\mathrm{perf}}/k} \rightrightarrows \pi_i\mathbb{L}^i_{R_{\mathrm{perf}}\otimes_R R_{\mathrm{perf}}\otimes_R R_{\mathrm{perf}}/k} \rightrightarrows \cdots$$

Each of the cotangent complexes appearing here is supported in degree i, so we may remove the π_i and instead shift by $[i]$; its totalisation is then precisely $\Omega^i_{R/k}[i]$, thanks to flat descent for $\mathbb{L}^i_{-/k}$.

To summarise, the coarse information of Lemma 3.8 is enough to show that $HP_0(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k)$ has a complete descending filtration with graded pieces $R[0]$, $\Omega^1_{R/k}[1]$, $\Omega^2_{R/k}[2]$,.... It follows quite formally[2] that $HP_0(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k)$ is necessarily equivalent to an actual complex of the form $R \to \Omega^1_{R/k} \to \Omega^2_{R/k} \to \cdots$, where each differential is the boundary map $H^i \to H^{i+1}$ induced by a fibre sequence coming from the filtration (see the footnote for details). But the complex is even a differential

[2] Here is a precise statement; for more details, including compatibility with symmetric monoidal structures, see [8, §5.1]:

Lemma 3.14. *Let R be a commutative ring and $C \in D(R)$ a complex equipped with a descending, $\mathbb{N}$-indexed filtration $C = \mathrm{Fil}^0 C \leftarrow \mathrm{Fil}^1 C \leftarrow \cdots$ such that, for each $i \ge 0$, the graded piece $\mathrm{gr}^i C$ is supported in degree i. Then C is naturally equivalent to the actual complex*

$$0 \longrightarrow H^0(\mathrm{gr}^0 C) \longrightarrow H^1(\mathrm{gr}^1 C) \longrightarrow H^2(\mathrm{gr}^2 C) \longrightarrow \cdots$$

where the boundary maps are the Bocksteins of the fibre sequences $\mathrm{gr}^{i+1} C \to \mathrm{Fil}^i C/\mathrm{Fil}^{i+2} C \to \mathrm{gr}^i C$; moreover, the filtration on C corresponds to the naive upwards truncations of this actual complex, and this process is compatible with symmetric monoidal structures.

Proof. By proving the result for each $C/\mathrm{Fil}^{n+1} C$ and then letting $n \to \infty$ we may suppose that the filtration on C is finite of length $n+1$, and by induction we may suppose that the result has been proved for any complex with a filtration of length n. Then we have a fibre sequence and equivalence

$$H^n(\mathrm{gr}^n C)[-n] = \mathrm{Fil}^n C \longrightarrow C \longrightarrow C/\mathrm{Fil}^n C \simeq [H^0(\mathrm{gr}^0 C) \to H^1(\mathrm{gr}^1 C) \to \cdots \to H^{n-1}(\mathrm{gr}^{n-1} C) \to 0],$$

whence C is the homotopy fibre of

$$[H^0(\mathrm{gr}^0 C) \to H^1(\mathrm{gr}^1 C) \to \cdots \to H^{n-1}(\mathrm{gr}^{n-1} C) \to 0] \xrightarrow{\beta} H^n(\mathrm{gr}^n C)[-n+1]$$

But β is precisely given by the desired Bockstein $H^{n-1}(\mathrm{gr}^{n-1} C) \to H^n(\mathrm{gr}^n C)$ and the homotopy fibre is given in the desired way. ☐

graded k-algebra since the above construction is compatible with product structures, so it is enough so check that $R \to \Omega^1_{R/k}$ is the usual de Rham differential. By picking an element of R and looking at the corresponding map $k[t] \to R$, functoriality reduces this to the case $R = k[t]$, in which case one computes explicitly to check it. □

3.4 The quasisyntomic site

This subsection can be omitted when reading the notes for the first time. In fact, in the current version of the notes it serves no purpose.

The arguments of §3.1–3.2 depended crucially not only on the cohomologiy theories of interest satisfying flat descent, but also on existence of the flat cover $R \to R_{\mathrm{perf}}$, for the smooth k-algebra R, having the following properties:

(a) All terms of the Cech resolution $R_{\mathrm{perf}} \otimes_R \cdots \otimes_R R_{\mathrm{perf}}$ are qrsp.

(b) The flat cover R_{perf} is functorial in R (to ensure naturality of the filtrations we constructed).

Although we argued directly to prove (a), it is also easy to check (we will do it in Lemma 3.16) that it is a consequence of $\mathbb{L}_{R_{\mathrm{perf}}/R}$ being supported in homological degree 1, where it is given by a flat R_{perf}-module. There do exist other rings admitting such a nicely behaved flat cover:

Definition 3.15. Given a morphism $A \to B$ of $\mathbb{F}_p$-algebras, we say that it is *quasisyntomic* if it is flat and $\mathbb{L}_{B/A}$ has cohomological Tor amplitude in $[-1, 0]$, i.e., for any B-module M, the complex $\mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B M$ is supported in cohomological degrees $[-1, 0]$. We say that it is a *quasisyntomic cover* if, in addition, it is faithfully flat.

We say that an $\mathbb{F}_p$-algebra A is *quasisyntomic* if $\mathbb{F}_p \to A$ is quasisyntomic in the above sense. Let QSyn denote the category of quasisyntomic $\mathbb{F}_p$-algebras (and all morphisms, not just quasisyntomic ones).

Quasisyntomic rings are precisely those which have a cover behaving similarly to $R \to R_{\mathrm{perf}}$:

Lemma 3.16. *An $\mathbb{F}_p$-algebra A is quasisyntomic if and only if there exist a qrsp $\mathbb{F}_p$-algebra S and a quasisyntomic cover $A \to S$. When this holds, all terms $S \otimes_A \cdots \otimes_A S$ of the Cech nerve are qrsp.*

To prove Lemma 3.16 it is helpful to first note the following general properties, which follow from the basic properties of the cotangent complex, especially transitivity (Proposition 2.27), so we leave the verifications to the reader:

Lemma 3.17. *(i) Let $A \to B$ be a quasisyntomic cover of $\mathbb{F}_p$-algebras; then A is quasisyntomic if and only if B is quasisyntomic.*

(ii) A composition of quasisyntomic morphism (resp. quasisyntomic covers) again a quasisyntomic morphism (resp. quasisyntomic cover).

(iii) A pushout of a quasisyntomic morphism (resp. quasisyntomic cover) along an arbitrary morphism is again a quasisyntomic morphism (resp. quasisyntomic cover).

Proof of Lemma 3.16. The implication $\Leftarrow$ is a consequence of Lemma 3.17(i). For the converse, suppose that A is quasisyntomic, let $\mathbb{F}_p[x_i : i \in I] \twoheadrightarrow A$ be a surjection from a polynomial algebra and let $\mathbb{F}_p[x_i^{1/p^\infty}] = \mathbb{F}_p[x_i : i \in I]_{\mathrm{perf}}$ be its colimit-perfection, which is easily seen to be a quasisyntomic cover of $\mathbb{F}_p[x_i]$; finally let $S := A \otimes_{\mathbb{F}_p[x_i]} \mathbb{F}_p[x_i^{1/p^\infty}]$ be the resulting pushout. Then $A \to S$ is the pushout of a quasisyntomic cover, hence is a quasisyntomic cover (by Lemma 3.17); finally note that S is semiperfect, since it is quotient of $\mathbb{F}_p[x_i^{1/p^\infty}]$.

To prove the Cech statement, let $A \to S$ be any quasisyntomic cover where S is qrsp. By taking repeated pushouts and compositions, Lemma 3.17(ii)&(iii) shows that $A \to S^{\otimes_A n}$ is quasisyntomic whence $S^{\otimes_A n}$ is quasisyntomic; but it is also clearly semiperfect since S is semiperfect. □

Remark 3.18. A stronger condition than quasisyntomic is *quasismooth*, which means that $A \to B$ is flat and $\mathbb{L}_{B/A}$ has Tor amplitude in $[0, 0]$, i.e., $\mathbb{L}_{B/A}$ is supported in degree 0 and $\Omega^1_{B/A}$ is a flat B-module. Just as in Lemma 3.17(ii)&(iii), compositions and pushouts of quasismooth maps are again quasismooth.

Example 3.19. If k is a perfect field (in fact, any perfect ring) of characteristic p and A is a smooth k-algebra (resp. perfect k-algebra), then A is quasismooth; if $I \subseteq A$ is a quasiregular ideal (e.g., generated by a regular sequence, then A/I is quasisyntomic). These are our main examples of interest, but the advantage of adopting an abstract definition in terms of the cotangent complex is that it avoids any finiteness hypothesis (which allows us to use, for example, polynomial rings in infinitely many variables, possibly over non-Noetherian perfect rings).

It follows from Lemma 3.17 that the opposite of QSyn is a site, with covers declared to be quasisyntomic covers. In other words, we have cut down the flat site on all $\mathbb{F}_p$-algebras by restricting to quasisyntomic algebras and by insisting that flat covers also be quasisyntomic. Lemma 3.16 gives us interesting covers with the same properties as (a), but without any naturality properties as in (b); next we explain the formalism which overcomes this lack of naturality.

Note that an $\mathbb{F}_p$-algebra A is quasiregular semiperfect if and only if it is quasisyntomic and semiperfect; so we write $\mathrm{QRSP} \subseteq \mathrm{QSyn}$ for the subcategory of qrsp algebras. The first assertion of the Lemma 3.16 implies in particular that $\mathrm{QRSP}^{\mathrm{op}}$ forms a basis for $\mathrm{QSyn}^{\mathrm{op}}$, as we showed that each object in the former site can be covered by an object in the latter. It follows that the categories of abelian sheaves on $\mathrm{QSyn}^{\mathrm{op}}$ and on $\mathrm{QRSP}^{\mathrm{op}}$ are equivalent via the obvious restriction functor. Using the second assertion of Lemma 3.16, this equivalence extends quite formally from abelian sheaves to ∞-sheaves taking values in any presentable ∞-category $\mathcal{D}$:

$$\mathrm{Sh}(\mathrm{QSyn}^{\mathrm{op}}, \mathcal{D}) \to \mathrm{Sh}(\mathrm{QRSP}^{\mathrm{op}}, \mathcal{D})$$

The key point is that, given an ∞-sheaf $\mathcal{F} : \mathrm{QRSP}^{\mathrm{op}} \to \mathcal{C}$, we may extend it to an ∞-sheaf $\mathcal{F}^{\sqsupset} : \mathrm{QSyn}^{\mathrm{op}} \to \mathcal{C}$ by setting

$$\mathcal{F}^{\sqsupset}(A) := \lim \big(\mathcal{F}(S) \rightrightarrows \mathcal{F}(S \otimes_A S) \rightrightarrows \mathcal{F}(S \otimes_A S \otimes_A S) \rightrightarrows \cdots \big),$$

where $A \in \mathrm{QSyn}$ and $A \to S$ is a chosen quasisyntomic cover with $S \in \mathrm{QRSP}$. The equivalence of ∞-categories assures us that this definition of $\mathcal{F}^{\sqsupset}$ is independent (up to equivalence) of the chosen cover $A \to S$, which is how we overcome the lack of the functorial cover $R \to R_{\mathrm{perf}}$ which we had in the smooth case. Following [8], the extension $\mathcal{F}^{\sqsupset}$ will be called the *unfolding* of $\mathcal{F}$.

In particular, suppose that $\mathcal{F}$ is a presheaf of abelian groups on $\mathrm{QSyn}^{\mathrm{op}}$ satisfying the following Cech condition:

> For any quasisyntomic cover $A \to B$ between qrsp algebras, the Cech complex $0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(B \otimes_A B) \to \cdots$ is exact, i.e., the higher Cech cohomology of $\mathcal{F}$ on A with respect to the cover B vanishes, and the 0^{th} Cech cohomology agrees with $\mathcal{F}(A)$. Or, in other words $\mathcal{F}|_{\mathrm{QRSP}}$, viewed as functor to $D(\mathbb{Z})$ is an ∞-sheaf on $\mathrm{QRSP}^{\mathrm{op}}$.

Then a general result about sites [Stacks Project, 03AN & 03F9] implies that $\mathcal{F}|_{\mathrm{QRSP}}$ is an abelian sheaf on $\mathrm{QRSP}^{\mathrm{op}}$ and that it has no higher (site-theoretic) cohomology on any object of $\mathrm{QRSP}^{\mathrm{op}}$. It follows also that the above Cech complex defining $(\mathcal{F}|_{\mathrm{QRSP}})^{\sqsupset}(A)$ is nothing other than the cohomology $R\Gamma_{\mathrm{QSyn}}(A, \mathcal{F})$ (we should more correctly write the sheafification $\mathcal{F}^{\mathrm{sh}}$ of $\mathcal{F}$ here instead, but we will usually lighten the notation by leaving the sheafification implicit when taking cohomology).

The previous paragraph applies in particular to our sheaves of interest:

Lemma 3.20. *The following abelian presheaves $\mathcal{F}$ on $\mathrm{QSyn}^{\mathrm{op}}$ all satisfy the previous Cech condition:*

(i) $\pi_n(\mathbb{L}^i_{-/\mathbb{F}_p})$ *for any* $i, n \geq 0$.

(ii) $HH_n(-/\mathbb{F}_p)$, $HC_n(-/\mathbb{F}_p)$, $HC^-_n(-/\mathbb{F}_p)$, $HP_n(-/\mathbb{F}_p)$ *for any* $n \in \mathbb{Z}$.

Proof. (i): If $i \neq n$ then $\pi_n(\mathbb{L}^i_{-/\mathbb{F}_p})$ is identically zero on QRSP by Remark 3.9, so it remains to deal with the case $i = n$. But then we have $\mathbb{L}_{-/\mathbb{F}_p} = \pi_1(\mathbb{L}_{-/\mathbb{F}_p})[1]$, so we must check that for any quasisyntomic cover $A \to B$ between qrsp algebras we have

$$\mathbb{L}^i_{A/\mathbb{F}_p} \xrightarrow{\sim} \lim \big(\mathbb{L}^i_{B/\mathbb{F}_p} \rightrightarrows \mathbb{L}^i_{B \otimes_A B/\mathbb{F}_p} \rightrightarrows \mathbb{L}^i_{B \otimes_A B \otimes_A B/\mathbb{F}_p} \rightrightarrows \cdots \big)$$

But this is exactly flat descent for the cotangent complex (Lemma 3.7).

(ii): As we saw at the start of Lemma 3.8, each $HH_n(-/\mathbb{F}_p)$ is either zero or given by $\pi_{n/2}(\mathbb{L}^{n/2}_{-/\mathbb{F}_p})$, to which we apply part (i). The Cech condition for the others then follows from the filtrations of Lemma 3.8. □

As a concrete example, the lemma and prior discussion mean that the Cech complex $HP_0(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k)$ which we studied in §3.2 can alternatively be described as

$$HP_0(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k) = R\Gamma_{\mathrm{QSyn}}(R, HP_{2i}(-/\mathbb{F}_p)),$$

the quasi-syntomic cohomology of (the sheafification of) $HP_0(-/k)$; and the same holds for $HC^-_n(-/\mathbb{F}_p)$, $HP_n(-/\mathbb{F}_p)$, and $HC_n(-/\mathbb{F}_p)$.

4 S^1-actions (three points of view) and *THH*

So far we have studied Hochschild and cyclic homology in terms of relatively explicit, but unmotivated, double complexes. In this section we will explain that there has been an action of the circle S^1 lurking in the background; although it may initially appear absurd to discuss S^1-actions on algebraic objects such as chain complexes, we will see several ways of making this precise. This S^1-action will play an essential role once we pass to topological cyclic homology, where the algebraic manifestations of this action will no longer be available.

First, in §4.1, we will explain that mixed complexes (Definition 2.19) may be viewed as complexes equipped with an S^1-action. Then in §4.2 we we drop any linearity assumptions and return to cyclic objects (Definition 2.17), which provide a different way of algebraically encoding an S^1-action. In §4.3 we will discover an even better (and our final) point of view, namely that of functors out of the classifying space of S^1. Once that formalism is available, we explain in §4.4 that the classical definition of Hochschild homology can be transported to any suitable symmetric monoidal (∞-)category, such as that of spectra; the result is topological Hochschild homology.

4.1 As $k[\varepsilon]/\varepsilon^2$-modules, i.e., mixed complexes

Given a fixed commutative base ring k, here we explain how mixed complexes over k provide a way of modelling what it means for a complex to be equipped with an S^1-action.

Note first that in the case of a discrete group G, a module M or complex with G-action is simply a module or complex over the group algebra $k[G]$; moreover, the group homology and cohomology are then given respectively by the derived coinvariants $k \otimes^{\mathbb{L}}_{k[G]} M$ and derived invariants $\mathrm{RHom}_{k[G]}(k, M)$.

Remark 4.1. The case in which G is a finite group is particularly interesting. Then the *norm map* $k \to k[G]$, $1 \mapsto \sum_{g\in G} g$ is well-defined and $k[G]$-linear (viewing k as a $k[G]$-algebra via the augmentation map), thereby inducing a natural transformation $k\otimes^{\mathbb{L}}_{k[G]} - \to \mathrm{RHom}_{k[G]}(k, -)$. Given a k-module M with G-action, the cofiber of this norm map $k \otimes^{\mathbb{L}}_{k[G]} M \to \mathrm{RHom}_{k[G]}(k, M)$ is known as the *Tate cohomology* (the same Tate cohomology which often appears, when G is a cyclic group, in class field theory).

Now we turn to the case of a topological group G. Possibly replacing G by a weakly equivalent topological group we may assume that G is the geometric realisation of a simplicial group $G_\bullet$ (for example, the Kan complex of simplices $\mathrm{Sing}(G)$ is a simplicial group whose geometric realisation is weakly equivalent to G, though in practice we will be interested in choosing some $G_\bullet$ which is much smaller than $\mathrm{Sing}(G)$). Then, in any category $\mathcal{C}$, we could define "an object with G-action" to be a simplicial object $X_\bullet$ of $\mathcal{C}$ which is equipped with an action by $G_\bullet$; i.e., each X_n is equipped with an action by G_n in a manner compatible with the face and degeneracy maps. In particular, if $X_\bullet$ were a simplicial set, or simplicial k-module, equipped with a $G_\bullet$-action in this sense, then its geometric realisation $|X_\bullet|$ would be a topological space equipped with a continuous action by $G = |G_\bullet|$ (simply because geometric realisation is functorial and compatible with products). Conversely, if X were any topological space equipped with a continuous action by $G = |G_\bullet|$, then $\mathrm{Sing}(X)$ would be equipped with an action by $\mathrm{Sing}(|G_\bullet|)$, which is equivalent to $G_\bullet$. In conclusion, up to issues of homotopy equivalence, actions by the simplicial group $G_\bullet$ offer a combinatorial model for actions by G. We will return to this point of view in §4.2, but now we wish to linearise by restricting to complexes of k-modules: then k-linear actions are modelled by modules over the simplicial k-algebra $k[G_\bullet]$, or equivalently connected differential graded modules over the dg k-algebra $Nk[G_\bullet]$ associated to the simplicial k-algebra $k[G_\bullet]$.

Remark 4.2. We have just used the following fact: given a simplicial ring $A_\bullet$, then its normalised cochain complex $NA_\bullet$ naturally admits the structure of a differential graded ring, with product given by the so-called "shuffle product". Note also that if $A_\bullet$ is commutative, then $NA_\bullet$ is strictly graded commutative.

We note also that passing from simplicial modules over $k[G_\bullet]$ to connective dg modules over $Nk[G_\bullet]$ does not lose any information: the two model categories are Quillen equivalent via the normalised complex construction [36, Thm. 1.1(2)].

We specialise now to the circle S^1, which is equivalent to the geometric realisation of $B\mathbb{Z}$, the classifying space construction on the infinite cyclic group:

$$B\mathbb{Z} = \qquad 0 \leftleftarrows \mathbb{Z} \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; \mathbb{Z}\times\mathbb{Z} \;\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\; \cdots$$

$$d_i(a_1,\dots,a_n) := \begin{cases} (a_2,\dots,a_n) & i=0 \\ (a_1,\dots,a_i+a_{i+1},\dots,a_n) & 0<i<n \\ (a_1,\dots,a_{n-1}) & i=n \end{cases} \qquad s_i(a_1,\dots,a_n) := (a_1,\dots,a_i,0,a_{i+1},\dots,a_n)$$

Lemma 4.3. *The dg k-algebra $Nk[B\mathbb{Z}]$ associated to the simplicial k-algebra $k[B\mathbb{Z}]$ is quasi-isomorphic to the dg k-algebra $k[\varepsilon]/\varepsilon^2$, where ε lies in homological degree 1 and the differential $k\varepsilon \to k$ is zero.*

Proof. Slick proof: since $B\mathbb{Z}$ is equivalent to $\mathrm{Sing}(S^1)$, the associated simplicial group algebra $k[B\mathbb{Z}]$ is equivalent to $\mathrm{Sing}(S^1;k)$, which in degree n is by definition the k-module generated by all n-simplices $\Delta^n \to S^1$. Passing to dgas, we must compute the usual dga computing $H_*(S^1,k)$, which is indeed $k[\varepsilon]/\varepsilon^2$.

Hands-on proof: the dga $Nk[B\mathbb{Z}]$ is the normalised complex $N_0 \xleftarrow{d_0} N_1 \xleftarrow{d_0} N_2 \xleftarrow{d_0} \dots$, where $N_n = \bigcap_{i=1}^n \mathrm{Ker}\, d_n \subseteq k[\mathbb{Z}^n] = k[t_1^{\pm1},\dots,t_n^{\pm1}]$. It is a good exercise in understanding definitions to check that this is indeed quasi-isomorphic to the desired $k[\varepsilon]/\varepsilon^2$. □

Therefore, once again up to issues of homotopy, we may view a connective complex C of k-modules as being equipped with an action by S^1 as soon as it is given the structure of a differential graded $k[\varepsilon]/\varepsilon^2$-module. Concretely, this means specifying a k-linear map $\varepsilon : C \to C[1]$ such that $\varepsilon^2 = 0$; but this is precisely the notion of a mixed complex from Definition 2.19! In fact, when mixed complexes were first introduced by Burghelea, they were called *algebraic S^1-complexes*.

Next we discuss cyclic and negative cyclic homology from this point of view. Observe that k, as a $k_\varepsilon := k[\varepsilon]/\varepsilon^2$-module, has a periodic resolution by free k_ε-modules

$$k \xleftarrow{\sim} k_{\mathrm{per}} := [k_\varepsilon \xleftarrow{\varepsilon} k_\varepsilon[1] \xleftarrow{\varepsilon} k_\varepsilon[2] \xleftarrow{\varepsilon} \cdots].$$

Given a mixed complex C over k, i.e., a connective dg $k[\varepsilon]/\varepsilon^2$-module, one easily checks the following:

- $HC(C/k) = [k_\varepsilon \xleftarrow{\varepsilon} k_\varepsilon[1] \xleftarrow{\varepsilon} \cdots] \otimes_{k[\varepsilon]/\varepsilon^2} C$, which represents $k \otimes^{\mathbb{L}}_{k[\varepsilon]/\varepsilon^2} C$, i.e., the "group homology of the action $S^1 \circlearrowright C$"
- $HC^-(C/k) = \mathrm{Hom}_{k[\varepsilon]/\varepsilon^2}(k_{\mathrm{per}}, C)$, which represents $\mathrm{RHom}_{k[\varepsilon]/\varepsilon^2}(k, C)$, i.e., the "group cohomology of the action $S^1 \circlearrowright C$"
- Moreover, the inclusion $k[1] = k\varepsilon \hookrightarrow k[\varepsilon]/\varepsilon^2$ is $k[\varepsilon]/\varepsilon^2$-linear and plays the role of the norm map from Remark 4.1; it induces $k \otimes^{\mathbb{L}}_{k[\varepsilon]/\varepsilon^2} C[1] \to \mathrm{Rhom}_{k_\varepsilon}(k, C)$, i.e., $HC(C/k)[1] \to HC^-(C/k)$. The reader should check that this map is precisely the boundary map in the "norm sequence" of Remark 2.15, whence its cofiber "Tate cohomology of the action $S^1 \circlearrowright C$" is $HP(C/k)$.

In conclusion, S^1-actions on connective chain complexes may be modelled as mixed complexes, in which case HC, HC^- and HP represent respectively group homology, cohomology, and Tate cohomology of the action.

4.2 As cyclic objects

Before restricting to the k-linear situation in §4.1, we observed that "an object with G-action" could be modelled as a simplicial object in the category $\mathcal{C}$ equipped with an action by the simplicial group $G_\bullet$ (where $G \simeq |G_\bullet|$). Specialising to the case of S^1, realised by the simplicial group $B\mathbb{Z}$, we could therefore study simplicial objects $X_\bullet$ equipped with an action by $B\mathbb{Z}$, i.e., each X_n should be equipped with n commuting automorphisms satisfying certain compatibilities with respect to the face and degeneracies. Unfortunately it seems that few concrete examples of such $B\mathbb{Z}$-actions exist; in particular, the Hochschild complex does not admit such an action on the nose (as far as the author is aware).

The problem is that although $B\mathbb{Z}$ is a simplicial group realising S^1, it is enormous when compared to the standard simplicial set S^1_{simp} realising S^1, whose only non-denerate simplices are one vertex and one 1-simplex:

$$S^1_{\text{simp}} = \qquad \{v\} \leftleftarrows \{v, e\} \,\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\, \{v, s_1e, s_0e\} \,\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\cdots$$

Unfortunately, S^1_{simp} is not a simplicial group (it is not even a Kan complex) and therefore it does not make sense to discuss its actions in the above sense. Remarkably, it does however admit the structure of a so-called *crossed simplicial group*, a notion introduced in the late 1980s by Krasauskas and Fiedorowicz–Loday [18] in order to provide much smaller simplicial models of certain topological groups by slightly weakening the concept of the simplicial group.

Definition 4.4. A *crossed simplicial group* is a simplicial set $\mathfrak{G}$ such that each $\mathfrak{G}_n$ is a group, together with group homomorphisms $\mathfrak{G}_n \to \operatorname{Aut}(\{0, \dots, n\})$, such that for all $0 \le i \le n$

(i) $d_i(gh) = d_i(g)d_{g^{-1}(i)}(h)$, $s_i(gh) = s_i(g)s_{g^{-1}(i)}(h)$, and

(ii) the following diagrams of sets commute

$$\begin{array}{ccc} [n-1] & \xrightarrow{d_{g^{-1}(i)}} & [n] \\ {\scriptstyle d_i(g)}\downarrow & & \downarrow{\scriptstyle g} \\ [n-1] & \xrightarrow[d_i]{} & [n] \end{array} \qquad \begin{array}{ccc} [n+1] & \xrightarrow{s_{g^{-1}(i)}} & [n] \\ {\scriptstyle s_i(g)}\downarrow & & \downarrow{\scriptstyle g} \\ [n-1] & \xrightarrow[s_i]{} & [n] \end{array}$$

An *action* of a crossed simplicial group $\mathfrak{G}_\bullet$ on a simplicial set $X_\bullet$ (or many generally on a simplicial object in an arbitrary category) is the data of actions of $\mathfrak{G}_n$ on X_n, for each $n \ge 0$, such that $d_i(gx) = d_i(g)(d_{g^{-1}(i)}(x))$ and $s_i(gx) = s_i(g)(s_{g^{-1}(i)}(x))$ for all $0 \le i \le n$.

Example 4.5. (i) Any simplicial group $G_\bullet$ may be viewed a crossed simplicial group, by declaring the representations $G_n \to \operatorname{Aut}(\{0, \dots, n\})$ to be trivial.

(ii) Connes' *cyclic* crossed simplicial group $\mathfrak{C}$ is defined by $\mathfrak{C}_n = \mathbb{Z}/n+1$, with the representation $\mathbb{Z}/n+1 \to \operatorname{Aut}(\{0, \dots, n\})$ being cyclic permutation and with simplicial structure isomorphic to S^1_{simp} given by the bijections

$$\mathbb{Z}/n+1 = \{0, \dots, n\} \cong S^1_{\text{simp},n}, \quad 0 \mapsto v, \quad 0 < i \mapsto s_{n-1}\cdots s_i s_{i-2}\cdots s_0(e).$$

It can be shown that the discussion after Remark 4.1 extends to crossed simplicial groups:

Proposition 4.6 ([18, §5]). *Let $\mathfrak{G}$ be a crossed simplicial group. Then:*

(i) Its geometric realisation $|\mathfrak{G}|$ naturally admits the structure of a topological group.

(ii) Given an action of $\mathfrak{G}$ on a simplicial set $X_\bullet$, there is an induced continuous action of $|\mathfrak{G}|$ on $|X_\bullet|$.

(iii) Adjoint: if X is a topological space equipped with a continuous action by $|\mathfrak{G}|$, then there is an induced action of $\mathfrak{G}$ on the simplicial set $\operatorname{Sing}(X)$.

Proof. The key observation is that, given any simpicial set $X_\bullet$, there exists a simplicial set $F_{\mathfrak{G}}(X_\bullet)$ with $\mathfrak{G}$-action which models the product $\mathfrak{G} \times X_\bullet$. More precisely, define $F_{\mathfrak{G}}(X_\bullet)$ as follows:

$$F_{G_\bullet}(X_\bullet)_n := \mathfrak{G}_n \times X_n, \qquad d_i(g,x) = (d_i(g), d_{g^{-1}(i)}(x)), \qquad s_i(g,x) = (s_i(g), s_{g^{-1}(i)}(x))$$

It is easy to check that $F_{\mathfrak{G}}(X_\bullet)$ admits a $\mathfrak{G}$ action (by the obvious left action of each $\mathfrak{G}_n$ on $\mathfrak{G}_n \times X_n$), and that there is even a monadic structure $F_{\mathfrak{G}}(F_{\mathfrak{G}}(X_\bullet)) \to F_{\mathfrak{G}}(X_\bullet)$, $(g,h,x) \mapsto (gh,x)$. Importantly, it can be shown that there is a natural homeomorphism of geometric realisations

$$|F_{\mathfrak{G}}(X_\bullet)| \overset{\simeq}{\to} |\mathfrak{G}| \times |X_\bullet|.$$

So, assuming now that $X_\bullet$ is equipped with a $\mathfrak{G}$-action, whence there is an action map $F_{\mathfrak{G}}(X_\bullet) \to X_\bullet$ (equivalently, $X_\bullet$ is an algebra over the monad $F_{\mathfrak{G}}$), passing to geometric realisations provides a continuous map $|\mathfrak{G}| \times |X_\bullet| \to |X_\bullet|$. Consideration of the special case $X_\bullet = \mathfrak{G}$ and the axioms for a monad easily proves (i) and (ii).

Part (iii) is slightly more involved so we refer the reader to [18, §5]. □

Remark 4.7. Part (i) of the proposition can be made more precise. For each $n \geq 0$, there is an inclusion of $\mathfrak{G}_n$ into $|\mathfrak{G}|$ as a discrete subgroup, given by $g \mapsto [g, (\frac{1}{n+1}, \ldots, \frac{1}{n+1})]$. When $n+1$ divides $n'+1$ these maps are moreover compatible with the degeneracy $[n'] \to [n]$ sending the first $\frac{n'+1}{n+1}$ elements to 0, the next $\frac{n'+1}{n+1}$ elements to 1, etc., so that $\mathfrak{G}_n \subseteq \mathfrak{G}_{n'} \subseteq |\mathfrak{G}|$. Passage to the limit identifies $\bigcup_{n\geq 0} \mathfrak{G}_n$ (a filtered union under the ordering $n+1|n'+1$) as a dense subgroup of $|\mathfrak{G}|$.

Thanks to Proposition 4.6 we see that, for any crossed simplicial group $\mathfrak{G}$, a combinatorial way to model $|\mathfrak{G}|$-actions is via $\mathfrak{G}$-actions. In particular, we may model $S^1 \simeq |\mathfrak{C}|$-actions by studying simplicial objects equipped with an action by Connes' cyclic group $\mathfrak{C}$. But, applying Definition 2.19 in the case of crossed simplicial group $\mathfrak{C}$, we see that an object with $\mathfrak{C}$-action is precisely a cyclic object in the sense of 2.17. In conclusion, cyclic objects in a category model S^1-actions.

Having heuristically claimed that both mixed complexes and cyclic objects model S^1-actions, we should compare the two approaches. As we saw in Section 2.2, the un-normalised chain complex construction defines a functor

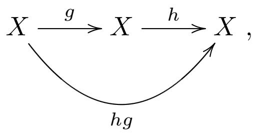
$$\{\text{cyclic } k\text{-modules}\} \longrightarrow \{\text{mixed complexes over } k\}$$

This functor is indeed an equivalence up to issues of homotopy (more precisely, it can be upgraded to a Quillen equivalence of model categories, or an equivalence of ∞-categories). We refer to [20, §II.2] [16, 17] [22, Corol. 2.4] for further details.

4.3 As functors $BS^1 \to \mathcal{C}$

Given a discrete group G, the data of an object of a category $\mathcal{C}$ equipped with a G-action is the same as giving a functor $G \to \mathcal{C}$, where we follow a standard abuse of notation of denoting by G the category with a single object $*$ having endomorphisms $\mathrm{End}_G(*) = G$. Given that the process of replacing a category by its nerve is fully faithful, this is then the same data as a morphism of simplicial sets (i.e., functor of ∞-categories) $BG = N(G) \to N(\mathcal{D})$.

However, let us now replace $N(\mathcal{D})$ by a homotopically richer category $\mathcal{D}$, such as a dg, simplicial, or ∞-category (in practice it will be $D(k)$ or Sp); we will begin to use some elementary language of ∞-categories (we refer the reader to §4.5 for a short introduction). Then a functor $BG \to \mathcal{D}$ is now slightly more subtle: it represents an object $X \in \mathcal{D}$, an auto-equivalence $g : X \overset{\sim}{\to} X$ for each $g \in G$, a 2-cell



for each $g, h \in G$, 3-cells connecting the 2-cells, and so on. That is, the action of G on X is no longer associative on the nose, but only up to higher homopoties in a coherent fashion.

This hopefully motives the following definition:

Definition 4.8. Given a topological group G, an *object of $\mathcal{D}$ with G-action* is a functor $BG \to \mathcal{D}$, where BG is a fixed Kan complex whose geometric realisation is a classifying space for G (as a topological group). The ∞-category of objects of $\mathcal{D}$ with G-action is thus $\mathcal{D}^{BG} := \mathrm{Fun}(BG, \mathcal{D})$.

Remark 4.9. (i) Any two choices of simplicial classifying space BG are homotopy equivalent, hence categorically equivalent as ∞-categories since they are Kan complexes, and therefore $\mathcal{D}^{BG}$ is well-defined up to categorical equivalence. If G is given to us as the geometric realisation of a simplicial group $G_\bullet$, then there are two common choices for BG, both built from the bisimplicial set $BG_\bullet$: either its diagonal or its $\overline{W}$-construction (which can be checked directly to be equivalent [?, ?]).

(ii) In the special case that $\mathcal{D} = N(\mathcal{C})$ is the nerve of a usual category $\mathcal{C}$, then the adjunction between nerves and homotopy categories shows that objects of $N(\mathcal{C})$ with G-action are simply objects of $\mathcal{C}$ with an action by the discrete group $\pi_0(G)$ in the usual sense. To get interesting examples it is essential that $\mathcal{D}$ be homotopically richer.

To compare Definition 4.8 to mixed complexes and cyclic objects we will consider two possible choices of simplicial classifying space BS^1.

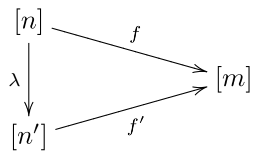
Firstly, it is well-known that the classifying space of S^1 is $\mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2) \simeq |BB\mathbb{Z}|$ [Mixed complex point of view to be added]

Our second model for the classifying space for S^1 comes from Connes' cyclic category:

Lemma 4.10. *The geometric realisation $|N(\Lambda)|$ of the cyclic category Λ is a classifying space for S^1.*

Proof. We will sketch one method of proof; for further details see [18, 5.9–5.12], for other approaches, see [32, App. T] [28].

One applies Quillen's Theorem B to the inclusion of categories $i : \Delta \hookrightarrow \Lambda$; recall that this is faithful and bijective on objects. For any object $[m] \in \Lambda$, the comma category $[m]\backslash i$ has objects given by all morphisms f in Λ with codomain $[m]$, and has morphisms given by commutative diagrams



where $\lambda \in \mathrm{Hom}_\Delta([n], [m]) \subseteq \mathrm{Hom}_\Lambda([n], [m])$. Thus its nerve $N([m]\backslash i)$ is the simplicial set $\mathrm{Hom}_\Lambda(-, [m])$. But any morphism $[n] \to [m]$ in Λ may be unique decomposed as an automorphism of $[n]$, i.e., an element of $\mathbb{Z}/n+1$, followed by a morphism $[n] \to [m]$ in Δ; this is part of the combinatorics which we omitted when defining the cyclic category. In other words, $\mathrm{Hom}_\Lambda([n], [m]) = \mathbb{Z}/n+1 \times \mathrm{Hom}_\Delta([n], [m])$; one then checks that the simplicial structure on the right side as n varies is precisely that which appeared in the construction $F_{\mathfrak{C}}$ in the proof of Proposition 4.6, i.e., $\mathrm{Hom}_\Lambda(-, [m]) = F_{\mathfrak{C}}(\Delta^m)$, where $\Delta^m := \mathrm{Hom}_\Delta(-, [m])$ is the m-simplex. [To finish; in the meantime, also for the details of the next paragraph, see [18]]

Quillen's Theorem B therefore implies that there is a homotopy fibre sequence

$$S^1 \simeq |\mathfrak{C}| \longrightarrow |N(\Delta)| \longrightarrow |N(\Lambda)|$$

But the geometric realisation of Δ is contractible (since, e.g., Δ has an initial object), and so $S^1 \simeq \Omega|N(\Lambda)|$. Taking care that this sufficiently preserves group structures (this is where we are omitting some details), one completes the proof by recalling that any space X serves as a classifying space for ΩX. □

The canonical unit map of simplicial sets $N(\Lambda) \to \mathrm{Sing}(|N(\Lambda)|)$ may therefore be written as $N(\Lambda) \to BS^1$, which by restriction induces a morphism

$$\mathcal{D}^{BS^1} \longrightarrow \mathrm{Fun}(N(\Lambda), \mathcal{D}),$$

i.e., each object X of $\mathcal{D}$ with S^1-action determines a cyclic object $X_\bullet$ of $\mathcal{D}$ (this is a simplicial object $X_\bullet$ of $\mathcal{D}$ equipped with $t_n : X_n \xrightarrow{\sim} X_n$ for each $n \geq 0$, such that the axioms of Definition 2.17 are satisfied up to higher homotopies in a coherent fashion). Informally, $X_\bullet$ is defined by taking the constant simplicial object $X_n := X$ for all n, with the cyclic operator t_n on X_n corresponding to the action of the cyclic group $\mathbb{Z}/n+1 \subseteq S^1$ on X.

We can also go the other direction (assuming that $\mathcal{D}$ admits geometric realisations of simplicial objects, which is true in cases like $D(k)$ or Sp): there is also a functor

$$\mathrm{Fun}(N(\Lambda), \mathcal{D}) \longrightarrow \mathcal{D}^{BS^1}$$

which takes a cyclic object $X_\bullet$ in $\mathcal{D}$ and gives $|X_\bullet| \in \mathcal{D}$ (the geometric realisation of the underlying simplicial object of $X_\bullet$) equipped with a certain natural S^1-action in the above sense. This is the essentially surjective right adjoint to the functor of the previous paragraph.

Remark 4.11. The previous two paragraphs are probably better known to the reader in the simpler case of Δ instead of Λ. Then $|N(\Delta)|$ is contractible and so the canonical unit map $N(\Delta) \to \mathrm{Sing}(|N(\Delta)|)$ may be written as $N(\Delta) \to *$. This induces the adjoint pair

$$\mathcal{D} = \mathrm{Fun}(*, \mathcal{D}) \underset{\text{geometric realisation}}{\overset{\text{constant simplicial object}}{\rightleftarrows}} \mathrm{Fun}(N(\Delta), \mathcal{D})$$

Given an object of $\mathcal{D}$, in general we do not care whether/how it arose as the geometric realisation of some simplicial diagram, so we work in the left category rather than the right.

Analogously to the remark, we do not care whether/how a given object of $\mathcal{D}^{BS^1}$ arose as the geometric realisation of some cyclic object. In conclusion, although we may pass through various constructions to get there, the final home for our objects with S^1-actions will be $\mathcal{D}^{BS^1}$.

4.4 The cyclic bar construction

Having hopefully justified in the section thus far that the correct framework in which to study S^1-actions is $\mathcal{D}^{BS^1}$, we should now offer an example. We will abstract the construction of Hochschild homology, rapidly leading to its topological analogue.

Recall that an *(associative) algebra object* in a symmetric monoidal category $(\mathcal{C}, \otimes, 1_\mathcal{C})$ is the data of an object $A \in \mathcal{C}$ together with morphisms $\mu : A \otimes A \to A$ (=multiplication) and $1_\mathcal{C} \to A$ (= multiplicative identity) which satisfy the usual commutative diagrams specifying the axioms of an algebra. Given such an algebra object A, one may write down a simplicial object

$$A \leftleftarrows A \otimes A \leftleftarrows A \otimes A \otimes A \leftleftarrows \cdots$$

in $\mathcal{C}$ by copying the rules for the face and degeneracy maps used in the definition of Hochschild homology from Remark 2.3. For example, the three face maps $d_0, d_1, d_2 : A \otimes A \otimes A \to A \otimes A$ are given by $\mu \otimes \mathrm{id}$, $\mathrm{id} \otimes \mu$, $\mu \otimes \mathrm{id} \circ t$, where t is the cyclic endomorphism of $A \otimes A \otimes A$ given by $a \otimes b \otimes c \mapsto c \otimes a \otimes b$. Moreover, by declaring t_n to be the analogous cyclic endomorphism of $A^{\otimes n+1}$ one upgrades this simplicial object to have the structure of a cyclic object in $\mathcal{C}$, known as the *cyclic bar construction* of A; it is sometimes denoted by $B^{\mathrm{cyc}}_\bullet(A)$.

We now wish to categorise this construction so that it may be applied to associative ring spectra, i.e., algebra objects for which the axioms of an algebra are only satisfied up to higher homotopties. The first step is to observe that algebra objects in $\mathcal{C}$ are classified by functors $\mathrm{Ass}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$ (Lemma 4.13 below), where the domain category is defined as follows:

Definition 4.12 (Associative algebras)**.** The category $\mathrm{Ass}^{\otimes}_{\mathrm{act}}$ is defined as follows: objects are finite sets, and a morphism from T to S is the data of a map of sets $f : T \to S$ together with linear orderings $\leq_s$ on the preimages $f^{-1}(s)$ for all $s \in S$. The composition of f with $g : S \to R$ is given by the set-theoretic composition $gf : T \to R$ together with the lexicographic orderings on the preimages; i.e., given $r \in R$ and $t, t' \in (gf)^{-1}(r)$, then $t <_r t'$ iff "$f(t) <_r f(t')$" or "$f(t) = f(t') =: s$ and $t <_s t'$".

The category $\mathrm{Ass}^{\otimes}_{\mathrm{act}}$ is symmetric monoidal under disjoint union, with unit element $\emptyset$. Note that the element $\{1\} \in \mathrm{Ass}^{\otimes}_{\mathrm{act}}$ admits the structure of an algebra object, with multiplication $\mu : \{1,2\} = \{1\} \sqcup \{1\} \to \{1\}$ corresponding to the ordering $1 < 2$. In fact, as promised, $\{1\} \in \mathrm{Ass}^{\otimes}_{\mathrm{act}}$ represents algebra objects in general:

Lemma 4.13. *The category of algebra objects in $\mathcal{C}$ (the notion of a morphism of algebra objects should be clear) identifies with the category of monoidal functors $F : \mathrm{Ass}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$, via $F \mapsto F(\{1\})$.*

Proof. A monoidal functor obviously sends algebra objects to algebra objects, so the functor $F \mapsto F(\{1\})$ with induced multiplication is indeed an algebra object in $\mathcal{C}$.

Conversely, from any algebra object $(A, \mu : A \otimes A \to A, 1_{\mathcal{C}} \to A)$, we may define a monoidal functor $A^{\otimes} : \mathrm{Ass}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$ by sending a finite set S to $A^{\otimes}(S) := \otimes_{s\in S} A$, and by sending a morphism $f : T \to S$ to

$$\otimes_{t\in T} A \longrightarrow \otimes_{s\in S} A, \qquad \otimes_{t\in T} a_t \mapsto \otimes_{s\in S} \prod_{t\in f^{-1}(s)} a_t.$$

Here we abusively write the product μ as though A were a set equipped with a law of multiplication, leaving the reader the write a categorical formula; the important point is that the product $\prod_{t\in f^{-1}(s)} a_t := a_{t_1} \cdots a_{t_m}$ is well-defined, since $f^{-1}(s) = \{t_1 <_s \cdots <_s t_m\}$ is equipped with a total order. ☐

The algebra object $\{1\} \in \mathrm{Ass}^{\otimes}_{\mathrm{act}}$ has an associated cyclic bar construction $B^{\mathrm{cryc}}_{\bullet}(\{1\})$

$$\{1\} \leftleftarrows \{1,2\} \,\overset{\leftarrow}{\leftleftarrows}\, \{1,2,3\} \,\overset{\leftleftarrows}{\leftleftarrows}\, \cdots.$$

This is a cyclic object in $\mathrm{Ass}^{\otimes}_{\mathrm{act}}$, so corresponds to a certain functor $\Lambda^{\mathrm{op}} \to \mathrm{Ass}^{\otimes}_{\mathrm{act}}$, which is denoted by Cut (and which can be described explicitly should one wish [32, App. T] [28]). Purely formally, it follows that the cyclic bar construction $B^{\mathrm{cryc}}_{\bullet}(A)$ of a general algebra object $A \in \mathcal{C}$ is now given by the functor

$$\Lambda^{\mathrm{op}} \xrightarrow{\mathrm{Cut}} \mathrm{Ass}^{\otimes}_{\mathrm{act}} \xrightarrow{A^{\otimes}} \mathcal{C},$$

where $A^{\otimes}$ is the functor corresponding to A in the sense of Lemma 4.13.

We are now prepared to extend the constructions to the ∞-world:

Definition 4.14. Let $\mathcal{D}$ be a symmetric monoidal infinity category. An *(associative) algebra object* in $\mathcal{D}$ is a functor $A : N(\mathrm{Ass}^{\otimes}_{\mathrm{act}}) \to \mathcal{D}$; note that this is the data of objects $A_1, A_2, \cdots \in \mathcal{D}$, various equivalences $A_n \otimes A_m \simeq A_{n+m}$, a multiplication map $A_2 \to A$, an identity map $1_{\mathcal{D}} \to A_1$, homotopies between them to express "associativity up to homotopy", etc. Also note that this is equivalent to the operadic definition of $\mathbb{E}_1$-algebras in $\mathcal{D}$ (see [28, Def./Prop. 3.3] and [31, Prop. 2.2.4.9]).

The *cyclic bar construction* $B^{\mathrm{cyc}}_{\bullet}(A)$ of an algebra object is now defined to be the functor

$$B^{\mathrm{cyc}}_{\bullet}(A) : N(\Lambda^{\mathrm{op}}) \xrightarrow{N(\mathrm{Cut})} N(\mathrm{Ass}^{\otimes}_{\mathrm{act}}) \xrightarrow{A} \mathcal{D},$$

which is simply making precise the statement there is a cyclic object in $\mathcal{D}$ which looks like

$$A \leftleftarrows A \otimes A \,\overset{\leftarrow}{\leftleftarrows}\, A \otimes A \otimes A \,\overset{\leftleftarrows}{\leftleftarrows}\, \cdots.$$

Finally, assuming that $\mathcal{D}$ admits geometric realisations of simplical objects, we may define the Hochschild homology of the algebra object A to be

$$HH(A/\mathcal{D}) := |B^{\mathrm{cyc}}_{\bullet}(A)| \in \mathcal{D}^{BS^1}$$

(recall from the end of §4.3 the geometric realisation of a cyclic object gives an object with S^1-action).

Thus $HH(-/\mathcal{D})$ is a (well-behaved) functor from the ∞-category of algebra objects in $\mathcal{D}$ to $\mathcal{D}^{BS^1}$.

Example 4.15. (i) Let k be a base ring and A a flat k-algebra. We leave it to the reader to write down a precise statement that Definition 4.14 recovers $HH(A/k)$.

(ii) More generally, if A is not necessarily a flat k-algebra, then it induces an algebra object A in the symmetric monoidal infinity category $D(k)$ such that $HH(A/D(k))$ is the derived form of $HH(A/k)$ introduced in §2.4 (since the monoidal structure in $D(k)$ is given by the derived tensor product $\otimes_k^{\mathbb{L}}$, which we modelled earlier by replacing A by a simplicial resolution $P_\bullet$).

(iii) Let A be an associate ring spectrum, i.e., an algebra object in Sp. Then its topological Hocshchild homology is

$$THH(A) := HH(A/\,\mathrm{Sp}) \in \mathrm{Sp}^{BS^1}.$$

We will discuss this further in the next subsection.

4.5 Topological Hochschild homology and its variants

A profound idea of Goodwillie and Waldhausen in the 1980s was to consider the formalism of Hochschild and cyclic homology not only for rings, but more generally for ring spectra. A well behaved theory of ring spectra was not available at the time (in particular, there was no strictly commutative and associative smash product, nor was the language of ∞-categories yet developed in order to overcome the higher coherence issues), but nevertheless Bökstedt succeeded in giving a rigorous definition of topological Hochschild homology (for "functors with smash product") in 1985 [10, 11]. Fortunately, the ∞-category of spectra is now available to us, and Nikolaus–Scholze have redeveloped the subject from this point of view. (Here we have omitted the entire history of the subject between 1985 and 2017, with apologies to all those concerned.)

We begin with an informal discussion of ∞-categories and spectra, for readers who have not previously encountered them. As is well-known, the triangulated category $D(k)$ of complexes of modules over a ring k up to quasi-isomorphism suffers from many problems, which can be resolved by keeping track of which quasi-isomorphisms were used to identify complexes. Of course, as soon as one keeps track of quasi-isomorphisms, one also needs to keep track of which homopoties between quasi-isomorphisms were used, and so on. The theory of ∞-categories (modelled as Joyal and Lurie's "quasi-categories") does this: one constructs a simplicial set $D(k)$ whose 0-simplices are chain complexes over k and whose higher simplices precisely encode the aforementioned data. To understand these notes, the reader should not run into too much trouble by treating ∞-categories as derived categories in which the usual problems of functoriality/uniqueness (of cones, homotopy limits, etc.) disappear, or rather become contractible in a precise sense. A good introduction to the foundation of the theory are the notes by Groth.

Next we turn to spectra. In the course of the historical development of mathematics, each new discovery of a rings of arithmetic interest was larger than the previous, i.e., $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$, but let us nevertheless consider the hypothetical situation of only knowing the complex numbers. This suffices to define complex manifolds, but in order to do Hodge theory we must first construct the integers, real numbers, and the functors

$$\mathbb{Z}\text{-mod} \underset{\text{restriction}}{\overset{-\otimes_{\mathbb{Z}}\mathbb{R}}{\rightleftarrows}} \mathbb{R}\text{-mod} \underset{\text{restriction}}{\overset{-\otimes_{\mathbb{R}}\mathbb{C}}{\rightleftarrows}} \mathbb{C}\text{-mod}$$

The faithful, non-full restriction functor $\mathbb{C}$-mod $\to$ $\mathbb{R}$-mod is particular important, as without it we would never see complex conjugation; in other words, it is essential to have the freedom to view our complex vector spaces merely as real vector spaces. Once we have done so, we can say that $\mathbb{R} \to \mathbb{C}$ is a morphism of algebra objects in $\mathbb{R}$-mod and thereby reconstruct $\mathbb{C}$-mod.

In a similar way, the theory of spectra provides a ring (in a generalised sense) $\mathbb{S}$ called the *sphere spectrum* and a restriction functor from $\mathbb{Z}$-modules to $\mathbb{S}$-modules which corresponds to a morphism of algebra objects $\mathbb{S} \to \mathbb{Z}$. Analogously to the naive Hodge theory example, it is now widely accepted that it is important to allow ourselves the freedom of viewing objects over $\mathbb{Z}$ as merely over $\mathbb{S}$: this allows for maps between them which are no longer $\mathbb{Z}$-linear, but merely $\mathbb{S}$-linear. Of course none of this is true in a naive set-theoretic fashion: only the derived category of $\mathbb{S}$-modules exists. So to be more precise there exist a symmetric monoidal ∞-category Sp of *spectra*, whose unit object is $\mathbb{S}$ and whose monidal structure is written $\otimes = \otimes_{\mathbb{S}}$, and functors

$$\mathrm{Sp} \underset{\text{restriction}}{\overset{-\otimes_{\mathbb{S}}\mathbb{Z}}{\rightleftarrows}} D(\mathbb{Z})$$

corresponding to a morphism of algebra objects $\mathbb{S} \to \mathbb{Z}$ in Sp. The restriction functor is known as the *Eilenberg–Maclane* construction, but we prefer not to introduce any notion for it and instead allow ourselves to view any complex as a spectrum. Any spectrum C has abelian "homotopy groups" $\pi_n(C)$, for $n \in \mathbb{Z}$, which are the usual homology groups $H_n(C)$ in case C comes via restriction from $D(\mathbb{Z})$ (or from $D(k)$ for some other ring k).

By restriction, any algebra object A in $D(\mathbb{Z})$ (e.g., a usual ring, or something more exotic like a differential graded ring) gives rise to an algebra object in Sp, i.e., a *ring spectrum*; we may then form its topological Hochschild homology as in Example 4.15(iii):

$$THH(A) := HH(A/\,\mathrm{Sp}) = |\, A \leftleftarrows A \otimes_{\mathbb{S}} A \leftleftarrows A \otimes_{\mathbb{S}} A \otimes_{\mathbb{S}} A \leftleftarrows \cdots | \in \mathrm{Sp}^{BS^1}$$

(Of course, as in Example 4.15(iii), we could also apply this construction to an arbitrary ring spectrum which does not necessarily come from $D(\mathbb{Z})$, but we are interested mainly in *THH* of usual rings.) The homotopy groups of this spectrum are denoted by $THH_n(A) := \pi_n THH(A)$ and are known as the *topological Hochschild homology groups* of A.

Remark 4.16 (Low degrees, rationalising, and stable homotopy groups of spheres)**.** The homotopy groups of the sphere spectrum $\mathbb{S}$ are given by

$$\pi_n(\mathbb{S}) = \begin{cases} 0 & n < 0 \\ \mathbb{Z} & n = 0 \\ n^{\text{th}} \text{ stable homotopy group of spheres} & n > 0 \end{cases}.$$

Two concrete consequences for the canonical map $THH(A) \to HH(A/\mathbb{Z})$ as are follows.

(i) It is an isomorphism in degrees ≤ 2, i.e., $THH_n(A) \overset{\simeq}{\to} HH_n(A/\mathbb{Z})$ for $n \geq 2$ (also both sides vanish for $n < 0$)

(ii) It is an isomorphism rationally (since the stable homotopy groups of spheres are known to be finite), or more precisely the map $THH_n(A) \to HH_n(A/\mathbb{Z})$ has kernel and cokernel killed by some integer depending only on n. In particular, if $A \supseteq \mathbb{Q}$ (whence the two sides are $\mathbb{Q}$-vector spaces), it is an isomorphism: topological Hochschild homology offers us nothing new in characteristic zero!

Let $\mathcal{D} = D(k)$ or Sp. Given any object with S^1-action $X \in \mathcal{D}^{BS^1}$, we may form its group cohomology, i.e., derived/homotopy S^1-invariants

$$X^{hS^1} = \lim_{BS^1} X \in \mathcal{D}$$

(where, on the right side, we recall that X is a functor $BS^1 \to \mathcal{D}$). Similarly we may form its group homology, i.e., derived/homotopy S^1-coinvariants

$$X_{hS^1} = \operatorname{colim}_{BS^1} X \in \mathcal{D}.$$

These are respectively the right and left adjoints to the functor $\mathcal{D} \to \mathcal{D}^{BS^1}$ which equips an object of $\mathcal{D}$ with the trivial action. There exists moreover a certain "norm" morphism $N : X_{hS^1}[1] \to X^{hS^1}$, whose cofiber is known as the *Tate cohomology* of the action and denoted by X^{tS^1}, i.e.,

$$X_{hS^1}[1] \xrightarrow{N} X^{hS^1} \longrightarrow X^{tS^1}.$$

Example 4.17. Suppose that the object with S^1-action in the preceding discussion is the Hochschild homology $HH(A/k) \in D(k)^{BS^1}$ of some k-algebra A. In §4.1 we informally explained that $HC(A/k)$,

$HC^-(A/k)$, and $HP(A/k)$ may be considered the homology, cohomology, and Tate cohomology of the S^1-action. To be precise, one does indeed have natural equivalences in $D(k)$

$$HC(A/k) \simeq HH(A/k)_{hS_1}, \qquad HC^-(A/k) \simeq HH(A/k)^{hS_1}, \qquad HP(A/k) \simeq HH(A/k)^{tS_1}.$$

These folklore identifications are obtained by picking particular models to compute the homotopy invariants and coinvariants, then comparing to the explicit double complexes of §2.2 (similarly to what we did at the end of §4.1); see [22].

Furthermore, if we rearrange the above norm sequence into a fibre sequence $X^{hS^1} \to X^{tS^1} \to X_{hS^1}[2]$, then the case $X = HH(A/k)$ recovers the sequence of Remark 2.15.

Definition 4.18. Transporting the previous definition to the world of spectra, we define the *topological negative cyclic homology* and *topological periodic cyclic homology* of a ring A to be

$$TC^-(A) := THH(A)^{hS^1} \in \mathrm{Sp}, \qquad TP(A) := THH(A)^{tS^1} \in \mathrm{Sp}\,.$$

We will also be interested in $THH(A)_{hS^1}$, which is an analogue of cyclic homology, but we stress that it is not what is known as topological cyclic homology. The homotopy groups of these spectra are denoted by $TC_n^-(A)$, $TP_n(A)$, $\pi_n THH(A)_{hS^1}$.

In the setting of the previous definition, the general norm sequence looks like

$$THH(A)_{hS^1}[1] \xrightarrow{N} TC^-(A) \longrightarrow TP(A).$$

Since $THH(A)_{hS^1}$ is supported in homotopical degree ≥ 0 (as this is true of A and preserved by colimits), we see that $TC_n^-(A) \overset{\cong}{\to} TP_n(A)$ for $n \leq 0$.

Remark 4.19 (Multiplicative structure)**.** Assuming that A is a commutative ring, which is once again our only case of interest, then $THH(A)$ is itself a ring spectrum. Moreover, taking the Tate construction or homotopy fixed points are known to be lax monoidal functors, so that $TP(A)$ and $TC^-(A)$ are ring spectra. Their homotopy groups

$$\bigoplus_{n\geq 0} THH_n(A), \qquad \bigoplus_{n\in\mathbb{Z}} TP_n(A), \qquad \bigoplus_{n\in\mathbb{Z}} TC_n^-(A)$$

therefore naturally have the structure of graded commutative rings.

5 Topological Hochschild homology of $\mathbb{F}_p$-algebras

In this section we restrict our study of topological Hochschild homology to the case of $\mathbb{F}_p$-algebras. We begin in §5.1 by studying $\mathbb{F}_p$ itself and deriving some analogues for general $\mathbb{F}_p$-algebras A; in particular, we will see that $TP(A)$ is a 2-periodic lifting of the classical periodic cyclic homology $HP(A/\mathbb{F}_p)$ from characteristic p to mixed characteristic, and establish an analogue of the HKR filtration of Proposition 2.28. We remark that these results do not have analogues for the topological Hochschild homology of arbitrary rings (though they do if we replace our base ring $\mathbb{F}_p$ by a perfectoid ring).

Then in §5.2 we analyse topological periodic and negative cyclic homologies of smooth algebras by reduction to qrsp algebras, analogously to §3.

5.1 The case of $\mathbb{F}_p$ itself and consequences

The following fundamental highly non-trivial result is sadly beyond the techniques of this course:

Theorem 5.1 (Bökstedt)**.** *The homotopy groups of $THH(\mathbb{F}_p)$ are given by*

$$THH_n(\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p & n \text{ even} \geq 0 \\ 0 & n \text{ otherwise} \end{cases}$$

Regarding multiplicative structure $THH_*(\mathbb{F}_p) = \mathbb{F}_p[u]$ (polynomial algebra on a single variable u) with $u \in THH_2(\mathbb{F}_p/\mathbb{Z}_p) = HH_2(\mathbb{F}_p/\mathbb{Z})$ (same element as in Example 2.32).

Although we have omitted the proof of Bökstedt's theorem, we will be able to use it as a blackbox to obtain interesting consequences, starting with a calculation of the homotopy groups of the topological periodic and negative cyclic homology of $\mathbb{F}_p$:

Proposition 5.2. *The homotopy groups of $TP(\mathbb{F}_p)$ are given by*

$$TP_n(\mathbb{F}_p) \cong \begin{cases} \mathbb{Z}_p & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Regarding multiplicative structure $TP_*(\mathbb{F}_p) = \mathbb{Z}_p[\sigma^{\pm 1}]$, where σ is any generator of the invertible $\mathbb{Z}_p$-module $TP_2(\mathbb{F}_p)$. The map of graded rings $TC^-_*(\mathbb{F}_p) \to TP_*(\mathbb{F}_p)$ is injective, with image

$$\mathrm{Im}(TC^-_{2n}(\mathbb{F}_p) \to TP_{2n}(\mathbb{F}_p)) = \begin{cases} p^n TP_{2n}(\mathbb{F}_p) & n \geq 0 \\ TP_{2n}(\mathbb{F}_p) & n \leq 0 \end{cases},$$

i.e., $TC^-_*(\mathbb{F}_p) = \mathbb{Z}_p[p\sigma, \sigma^{-1}]$, and the canonical morphism $TC^-(\mathbb{F}_p) \to THH(\mathbb{F}_p)$ sends $p\sigma$ to u (equivalently, without making any choices, $TC^-_*(\mathbb{F}_p)/p \overset{\simeq}{\to} THH_*(\mathbb{F}_p)$ for $* \geq 0$).

Remark 5.3. It may be helpful to compare to Lemma 3.8 and Remark 3.10, where HP_0 inherited an interesting filtration coming from HC^-_{2n}, $n \geq 0$. Here we witness the same phenomenon: the homotopy groups of $TC^-(\mathbb{F}_p)$ is inducing the p-adic filtration on the homotopy groups of $TP(\mathbb{F}_p)$.

Proof of Proposition 5.2. Recall that there is a class $u \in THH_2(\mathbb{F}_p)$ which corresponds to $p \in p\mathbb{Z}/p^2\mathbb{Z}$ under the identifications $THH_2(\mathbb{F}_p) \overset{\simeq}{\to} HH_2(\mathbb{F}_p/\mathbb{Z}) = p\mathbb{Z}/p^2\mathbb{Z}$. We will also use the cohomology class $t \in H^2(S^1, \mathbb{F}_p)$ which corresponds to the extension $0 \to \mathbb{F}_p \to S^1 \overset{p}{\to} S^1 \to 0$; a standard fact about group cohomology states that $H^*(S^1, \mathbb{F}_p) \cong \mathbb{F}_p[t]$.

Using Bökstedt's theorem and the previous paragraph, the homotopy-fixed points spectral sequence for the action of S^1 on $THH(\mathbb{F}_p)$ looks as follows and converges to $TC^-_{i+j}(\mathbb{F}_p)$:

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots\, 0 & 0 & 0 & 0 & 0 \\
\cdots\, H^4(S^1, THH_2(\mathbb{F}_p)) = \mathbb{F}_p t^2 u & 0 & H^2(S^1, THH_2(\mathbb{F}_p)) = \mathbb{F}_p t u & 0 & H^0(S^1, THH_2(\mathbb{F}_p)) = \mathbb{F}_p u \\
\cdots\, 0 & 0 & 0 & 0 & 0 \\
\cdots\, H^4(S^1, THH_0(\mathbb{F}_p)) = \mathbb{F}_p t^2 & 0 & H^2(S^1, THH_0(\mathbb{F}_p)) = \mathbb{F}_p t & 0 & H^0(S^1, THH_0(\mathbb{F}_p)) = \mathbb{F}_p
\end{array}$$

(the bottom right corner is in bidegree $(0,0)$). Since all non-zero terms are in even bidegree, all differentials on all pages are necessarily zero, i.e., the spectral sequence is degenerate.

The key (in fact, only) non-trivial calculation is to check the following: the usual spectral sequence map

$$\mathrm{Fil}^2_{\text{abutement}}\, TC^-_0(\mathbb{F}_0) = \mathrm{Ker}\left(TC^-_0(\mathbb{F}_p) \xrightarrow{\text{edge}} H^0(S^1, THH_0(\mathbb{F}_p))\right) \longrightarrow H^2(S^1, THH_2(\mathbb{F}_p))$$

sends p to tu. This is proved as follows: the part of the spectral sequence which we have displayed (i.e., involving $THH_n(\mathbb{F}_p)$ for $n \leq 2$) is the same as the analogous spectral sequence for the Hochschild homology $HH(\mathbb{F}_p/\mathbb{Z})$, and therefore the calculation is a claim purely about usual Hochschild and negative cyclic homology. One resolves $\mathbb{F}_p$ by a flat dg $\mathbb{Z}$-algebra and proceeds by a direct calculation, for which we refer to [28, Prop. 2.12].

By multiplicativity of the spectral sequence it follows that $p^i \in \mathrm{Fil}^{2i}_{\mathrm{abut}} TC_0^-(\mathbb{F}_p)$ for all $i \geq 0$, and that the canonical map $\mathrm{Fil}^{2i}_{\mathrm{abut}} TC_0^-(\mathbb{F}_p) \to H^{2i}(S^1, THH_{2i}(\mathbb{F}_p))$ sends p^i to $t^i u^i$. That is, the unique ring homomorphism $\mathbb{Z} \to TC_0^-(\mathbb{F}_p)$ induces isomorphism $p^i\mathbb{Z}/p^{i+1}\mathbb{Z} \xrightarrow{\simeq} \mathrm{gr}^{2i}_{\mathrm{abut}} TC_0^-(\mathbb{F}_p)$ for all $i \geq 0$. It follows that $\mathbb{Z}_p \xrightarrow{\simeq} TC_0^-(\mathbb{F}_p)$.

Pick lifts $u \in TC_2^-(\mathbb{F}_p)$ and $t \in TC_{-2}^-(\mathbb{F}_p)$, along the edge maps, of u and t respectively (it would be more correct to write $\widetilde{u}$ and $\widetilde{t}$ for the lifts, but we follow a standard abuse of notation). The multiplicativity of the spectral sequence forces $TC_{2i}^-(\mathbb{F}_p) = \mathbb{Z}_p u^i$ for $i \geq 0$, and $TC_{2i}^-(\mathbb{F}_p) = \mathbb{Z}_p t^i$ for $i \leq 0$, with the relation $ut = p \cdot \text{unit}$ in $TC_0^-(\mathbb{F}_p) = \mathbb{Z}_p$. □

Remark 5.4. Theorem 5.1 and Proposition 5.2 remain true if $\mathbb{F}_p$ is replaced by an arbitrary perfect $\mathbb{F}_p$-algebra k and $\mathbb{Z}_p$ is replaced by the ring of p-typical Witt vectors $W(k)$.

As we promised earlier, from the proposition one obtains some interesting general consequences which will allow us to control the topological theory in characteristic p. The first is that topological periodic cyclic homology of $\mathbb{F}_p$-algebras is always 2-periodic; despite its name, this is not always the case, notably $TP(\mathbb{Z})$. The second is that it lifts classical periodic cyclic homology; since we already know that the latter is related to de Rham cohomology, this indicates that the former might be related to the standard lift of de Rham cohomology, namely crystalline cohomology (as we shall see is indeed true).

Corollary 5.5. *Let A be a commutative $\mathbb{F}_p$-algebra. Then:*

(i) There is a natural, S^1-equivariant fibre sequence $THH(A)[2] \xrightarrow{u} THH(A) \to HH(A/\mathbb{F}_p)$.
(This will let us control THH in terms of HH and so serve as a replacement for the classical periodicity sequence.)

(ii) $TP(A)$ is 2-periodic, i.e., the $TP_0(A)$-module $TP_2(A)$ is invertible and $TP_n(A) \otimes_{TP_0(A)} TP_2(A) \xrightarrow{\simeq} TP_{n+2}(A)$ for all $n \in \mathbb{Z}$.

(iii) There is a natural equivalence $TP(A)/p \to HP(A/\mathbb{F}_p)$.

Proof. We need the following general result: given a map of commutative rings $k \to A$, then in the world of $\mathbb{E}_\infty$-ring spectra one has $THH(A) \otimes_{THH(k)} k \simeq HH(A/k)$. This is proved either by checking that both sides satisfy an identical universal property similar to that of Example 2.18, or by transferring the proof of Remark 2.7 to the $\mathbb{E}_\infty$ context.

To prove (i), it is therefore enough to construct an S^1-equivariant fibre sequence

$$THH(\mathbb{F}_p)[2] \to THH(\mathbb{F}_p) \to \mathbb{F}_p,$$

since we can then base change by $THH(A) \otimes_{THH(\mathbb{F}_p)} -$. The class $u := p\sigma \in TC_2^-(\mathbb{F}_p) = \mathrm{Hom}_{h(\mathrm{Sp})}(\mathbb{S}[2], TC^-(\mathbb{F}_p))$ corresponds to a morphism $u : \mathbb{S}[2] \to TC^-(\mathbb{F}_p)$ uniquely up to homotopy; composing with $TC^-(\mathbb{F}_p) \to THH(\mathbb{F}_p)$ gives us a S^1-equivariant map $\mathbb{S}[2] \to THH(\mathbb{F}_p)$, which then linearises to a morphism $u : THH^-(\mathbb{F}_p)[2] \to THH^-(\mathbb{F}_p)$ whose effect on homotopy groups is multiplication by x thanks to the final sentence of Proposition 5.2. We can also read off that proposition that the cofiber of $\times u$ is $\mathbb{F}_p[0]$, as desired.

(ii) follows formally from the fact that $TP_*(A)$ is a graded module over $TP_*(\mathbb{F}_p) \cong \mathbb{Z}_p[\sigma^{\pm 1}]$.

(iii): Taking Tate constructions in the fibre sequence of (i) gives

$$TP(A)[2] \xrightarrow{u} TP(A) \to HP(A/\mathbb{F}_p)$$

But by construction u is given by

$$\begin{array}{ccc} TP(A)[2] & \xrightarrow{\quad u \quad} & TP(A) \\ & {\scriptstyle \sigma,\ \cong} \searrow \quad \nearrow {\scriptstyle p} & \\ & TP(A) & \end{array}$$

□

Suppose now that R is a smooth algebra over a perfect field k of characteristic p (in fact, we could suppose that k is any perfect $\mathbb{F}_p$-algebra without any change to the argument). Then we have the anti-symmstrisation map $\Omega^1_{R/\mathbb{F}_p} = \Omega^1_{R/\mathbb{Z}} \to HH_1(R/\mathbb{Z}) = THH_1(R)$ which we know induces by multiplication $\Omega^2_{R/\mathbb{F}_p} \to HH_2(R/\mathbb{Z}) = THH_2(R)$; since $THH_*(R)$ is a commutative graded ring, multiplicativity formally implies that the antisymmstrisation map extends to all degrees $\Omega^*_{R/k} \to THH_*(R)$. Combining this with the map of graded rings $THH_*(\mathbb{F}_p) = \mathbb{F}_p[u] \to THH_*(R)$, we arrive at $\Omega^*_{R/k} \otimes_{\mathbb{F}_p} THH_*(\mathbb{F}_p) \to THH_*(R)$, where $\otimes$ denotes tensor product of graded rings. One has the following analogue of the classical HKR theorem (it may initially appear strange that *THH* is built from multiple copies of de Rham groups, but when passing to *TP* these will perfectly stack up on top of each other to form cristalline cohomology):

Proposition 5.6 (Hesselholt's HKR theorem)**.** *Let R be a smooth k-algebra as in the previous paragraph. Then the map of graded rings $\Omega^*_{R/k} \otimes_{\mathbb{F}_p} THH_*(\mathbb{F}_p) \to THH_*(R)$ is an isomorphism. In particular, $THH_n(R)$ is isomorphic as an R-module to $\bigoplus_{i\geq 0} \Omega^{n-2i}_{R/k}$.*

Proof. By construction the composition

$$\Omega^*_{R/k} \subseteq \Omega^*_{R/k} \otimes_{\mathbb{F}_p} THH_*(\mathbb{F}_p) \to THH_*(R) \to HH_*(R)$$

is the usual antisymmetrisation map, which is an isomorphism by the usual HKR theorem. So $THH_*(A) \to HH_*(A/\mathbb{F}_p)$ is surjective on homotopy groups and the fibre sequence of Corollary 5.5(i) breaks into short exact sequence, whence we have an isomorphism of graded rings $THH_*(A)/u \xrightarrow{\simeq} HH_*(A/\mathbb{F}_p)$.

In other words, the map of graded $\mathbb{F}_p[x]$-modules

$$\Omega^*_{R/k} \otimes_{\mathbb{F}_p} THH_*(\mathbb{F}_p) = \Omega^*_{R/k} \otimes_{\mathbb{F}_p} \mathbb{F}_p[x] \longrightarrow THH_*(R)$$

is an isomorphism modulo x. It follows formally (by induction up the degree) that it is an isomorphism. □

Hesselholt's HKR theorem leads to an analogue of the HKR filtration from Proposition 2.28, which gives us a technique to control *THH* via the cotangent complex.

Corollary 5.7 (HKR filtration on *THH*)**.** *Let A be an $\mathbb{F}_p$-algebra. Then $THH(A)$ admits a descending, $\mathbb{N}$-indexed, complete filtration whose n^{th}-graded piece is $\bigoplus_{i=0}^{\lfloor n/2 \rfloor} \mathbb{L}^{n-2i}_{A/\mathbb{F}_p}[n]$.*

Proof. Just left Kan extend the previous proposition; see [8, Corol. 6.10] for some details. □

In order to extend the techniques of §3.1–3.2 to topological Hochschild homology and its variants, we of course need to know that they satisfy flat descent; although there is a technique to deduce this in general from the case of *HH* (see [8, Corol. 3.3], the following special case is enough for us:

Corollary 5.8. *The* Sp*-valued functors*

$$THH(-), \qquad THH(-)_{hS^1}, \qquad TC^-(-), \qquad TP(-)$$

on $\mathbb{F}_p$-algs *satisfy flat descent.*

Proof. The argument is similar to Lemma 3.7. The previous corollary and flat descent for all the $\mathbb{L}^i_{-/\mathbb{F}_p}$ yield flat descent for $THH(-)$. The other cases then follow by suitably taking limits and colimits, taking care of the connectivity issues which arise for $THH(-)_{hS^1}$. □

5.2 Topological periodic & negative cyclic homology of smooth and qrsp algebras

The main theorem we wish to discuss is the following analogue of Theorem 3.1 in which *HP* is replaced by *TP* and de Rham cohomology is replaced by crystalline cohomology:

Theorem 5.9. *If R is a smooth k-algebra, where k is a perfect field of characteristic $p > 0$, then $TP(R)$ and $TC^-(R)$ admit natural, complete, descending $\mathbb{Z}$-indexed filtrations whose n^{th} graded pieces are given respectively by*

$$R\Gamma_{\text{crys}}(R/W(k))[2n], \qquad \mathcal{N}^{\geq i} R\Gamma_{\text{crys}}(R/W(k))[2n],$$

where $\mathcal{N}^{\geq i}$ refers to a certain filtration on crystalline cohomology known as the Nygaard filtration.

The filtrations of the theorem are precisely those arising from the general technique presented in Corollary 3.6, which is valid since we verified flat descent for TP and TC^- in Corollary 5.8. Therefore, following the same method of proof of §3.1–3.2, we must attempt to identify TP and TC^- of the qrsp rings $R_{\text{perf}} \otimes_R \cdots \otimes_R R_{\text{perf}}$. Let we begin with some coarse information analogous to Lemma 3.8

Lemma 5.10. *Let A be a qrsp algebra and set $N := \pi_1(\mathbb{L}_{A/\mathbb{F}_p})$. Then $THH_*(A)$, $TC^-_*(A)$, $TP_*(A)$, and $\pi_*THH(A)_{hS^1}$ are all supported in even degrees. Moreover, $THH_{2i}(A)$ admits a finite increasing filtration with graded pieces $A, N, \Gamma^2_A(N), \dots, \Gamma^i_A(N)$, and $TP_0(A)$ admits a complete, descending, $\mathbb{N}$-indexed filtration by ideals such that*

$$\text{Fil}^i \cong TC^-_{2i}(A), \qquad TP_0(A)/\,\text{Fil}^i \cong \pi_{2i-2}THH(A)_{hS^1}, \qquad \text{gr}^i \cong THH_{2i}(A) \cong \pi_i(\mathbb{L}^i_{A/\mathbb{F}_p})$$

for all $i \geq 0$.

Proof. The proof is entirely analogous to that of Lemma 3.8; firstly, as already argued there, each wedge powers $\mathbb{L}^i_{A/\mathbb{F}_p} = \Gamma^i_A(N)[i]$ is supported in degree i. The HKR filtration of Corollary 5.8 therefore proves that $THH_*(A)$ is supported in even degree and gives the desired finite increasing filtration (we leave it to the reader to carefully check that the degrees are ok).

The rest of the structure can be formally read off Corollary 5.5 and the norm sequence $THH(A)_{hS^1} \to TC^-(A) \to TP(A)$, in a very similar way to the arguments of Lemma 3.8. □

Continuing to following the same line of argument we present in §3.2, we would now like to identify explicitly the filtered ring $TP_0(A)$. In the case of classical periodic cyclic homology we could do this in two ways, either in terms of a divided power envelope or in terms of derived de Rham cohomology. Now we lift these two constructions from characteristic p to mixed characteristic.

5.2.1 First construction: derived de Rham(–Witt) and crystalline cohomology

Let R be a smooth algebra over a perfect field k of characteristic p, and let $\widetilde{R}$ be a p-adically complete, formally smooth $W(k)$-algebra lifting R (i.e., $\widetilde{R}/p = R$). Let $\Omega^\bullet_{\widetilde{R}/W(k)}$ be the p-adically complete de Rham complex (we should really add a hat to indicate the p-adic completion, but we prefer to keep notation light and will never consider the non-complete version).

Assuming that $\widetilde{R}$ is equipped with a lift of the absolute Frobenius (i.e., there exists a ring homomorphism $\widetilde{\varphi} : \widetilde{R} \to \widetilde{R}$ compatible with the usual Frobenius on $W(k)$ and such that $\widetilde{\varphi}(f) \equiv f^p \bmod p\widetilde{R}$ for all $f \in \widetilde{R}$), then we define a filtration on $\Omega^\bullet_{\widetilde{R}/W(k)}$ by setting, for $i \geq 0$,

$$p^{\max(i-\bullet,0)}\Omega^\bullet_{\widetilde{R}/W(k)} := \qquad p^i\widetilde{R} \xrightarrow{d} p^{i-1}\Omega^1_{\widetilde{R}/W(k)} \xrightarrow{d} \cdots \xrightarrow{d} p\Omega^{i-1}_{\widetilde{R}/W(k)} \xrightarrow{d} \Omega^i_{\widetilde{R}/W(k)} \xrightarrow{d} \Omega^{i+1}_{\widetilde{R}/W(k)} \xrightarrow{d} \cdots$$

(although the definition of this filtration does not depend on $\widetilde{\varphi}$, it is only reasonable to define it when $\widetilde{\varphi}$ exists; this is partly because of the next lemma).

Lemma 5.11. *The graded pieces of the above filtration on $\Omega^\bullet_{\widetilde{R}/W(k)}$ are given (up to quasi-isomorphism) by*

$$\widetilde{\varphi}/p^i : \text{gr}^i_{\mathcal{N}} R\Gamma_{\text{crys}}(R/W(k)) \xrightarrow{\sim} \tau^{\leq i}\Omega_{R/k}$$

(where $\tau^{\leq i}$ denotes canonical – not naive – truncation).

Proof. Note that $\widetilde{\varphi}$ induces an endomorphism of $\Omega^n_{\widetilde{R}/W(k)}$ which is divisible by p^n, so there is indeed an induced map as indicated. The graded pieces of the filtration are given by $R \xrightarrow{0} \Omega^1_{R/k} \xrightarrow{0} \cdots \xrightarrow{0} \Omega^i_{R/k}$ and the induced map on cohomology looks like $\Omega^n_{R/k} \to H^n_{\text{dR}}(R/k)$ (for $n \leq i$); one checks that it is precisely the Cartier isomorphism. □

As an object of the derived category $D(W(k))$, the complex $\Omega^\bullet_{\widetilde{R}/W(k)}$ depends only on R. Indeed, the theory of crystalline cohomology implies that $\Omega^\bullet_{\widetilde{R}/W(k)}$ is equivalent to the crystalline cohomology $R\Gamma_{\mathrm{crys}}(R/W(k))$, or equivalently to the de Rham–Witt complex $W\Omega^\bullet_{R/k}$. This independence remains true for the filtration we have just defined:

Lemma 5.12. *The above filtration on $\Omega^\bullet_{\widetilde{R}/W(k)}$ also depends only on R. (More precisely $\Omega^\bullet_{\widetilde{R}/W(k)}$, as an object of the filtered derived category over $W(k)$, depends naturally on R.)*

Proof. There are two ways to check this, either via crystalline cohomology or via de Rham–Witt theory. For the crystalline approach one uses a result of Berthelot–Ogus, stating that the Frobenius φ induces an equivalence $\varphi : R\Gamma_{\mathrm{crys}}(R/W(k)) \xrightarrow{\sim} L\eta_p R\Gamma_{\mathrm{crys}}(R/W(k))$, where $L\eta_p$ refers to the décalage functor of [8]. There is a natural filtration on the right side arising from the décalage functor (see ...), which therefore induces a filtration on the left side. Under the equivalence $\Omega^\bullet_{\widetilde{R}/W(k)} \simeq \Gamma_{\mathrm{crys}}(R/W(k))$, this is precisely the above filtration on $\Omega^\bullet_{\widetilde{R}/W(k)}$.

For the de Rham–Witt approach, recall that $\Omega^\bullet_{\widetilde{R}/W(k)}$ identifies up to equivalence with $W\Omega^\bullet_{R/k}$; then the above filtration can be shown to identify with the so-called *Nygaard filtration* on $W\Omega^\bullet_{R/k}$ given by

$$\mathcal{N}^{\geq i} W\Omega^\bullet_{R/k} := \quad p^{i-1}VW(R) \xrightarrow{d} p^{i-2}VW\Omega^1_R \xrightarrow{d} \cdots \xrightarrow{d} pVW\Omega^{i-2}_R \xrightarrow{d} VW\Omega^{i-1}_R \xrightarrow{d} W\Omega^i_R \xrightarrow{d} W\Omega^{i+1}_R \xrightarrow{d} \cdots .$$

□

In conclusion, we have naturally associated to each smooth k-algebra R a filtered complex $R\Gamma_{\mathrm{crys}}(R/W(k))$ (even an $\mathbb{E}_\infty$-$W(k)$-algebra). If we left Kan extend and complete with respect to the resulting filtration then the result, in the case of a qrsp algebra A, is a complete filtered ring $\widehat{\mathbb{L}W\Omega}_{A/\mathbb{F}_p}$ such that $\widehat{\mathbb{L}W\Omega}_{A/\mathbb{F}_p}/p = \widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p}$. More details to be added.

5.2.2 Second construction: a divided power envelope

Let A be qrsp. We denote by $\mathbb{A}^\circ_{\mathrm{crys}}(A)$ the divided power envelope of the composition $W(A^\flat) \twoheadrightarrow A^\flat \twoheadrightarrow A$, and by $\mathbb{A}_{\mathrm{crys}}(A)$ its p-adic completion. For example, $\mathbb{A}_{\mathrm{crys}}(\mathcal{O}_{\overline{\mathbb{Q}}_p}/p)$ is the period ring $\mathbb{A}_{\mathrm{crys}}$ of p-adic Hodge theory.

The usual Witt vector Frobenius φ induces a Frobenius endomorphism φ of $\mathbb{A}_{\mathrm{crys}}(A)$, using which we define its *Nygaard filtration* as follows:

$$\mathcal{N}^{\geq i}\mathbb{A}_{\mathrm{crys}}(A) := \{f \in \mathbb{A}_{\mathrm{crys}}(A) : \varphi(f) \in p^i \mathbb{A}_{\mathrm{crys}}(A)\}.$$

Finally, write $\widehat{\mathbb{A}}_{\mathrm{crys}}(A) := \varprojlim_{i\to\infty} \mathbb{A}_{\mathrm{crys}}(A)/\mathcal{N}^{\geq i}\mathbb{A}_{\mathrm{crys}}(A)$ for the completion with respect to the Nygaard filtration, and $\widehat{\mathbb{A}}_{\mathrm{crys}}(A) := \mathrm{Ker}(\widehat{\mathbb{A}}_{\mathrm{crys}}(A) \twoheadrightarrow \mathbb{A}_{\mathrm{crys}}(A)/\mathcal{N}^{\geq i}\mathbb{A}_{\mathrm{crys}}(A))$ for the induced filtration.

The advantage of $\widehat{\mathbb{A}}_{\mathrm{crys}}(A)$ is that it is reasonably explicit.

5.2.3 Back to *TP*

The following is the analogue for the topological theories of Proposition 3.13:

Proposition 5.13. *Let A be a qrsp algebra. Then there are natural isomorphisms of filtered rings*

$$TP_0(A) \cong \widehat{\mathbb{A}}_{\mathrm{crys}}(A) \cong \widehat{\mathbb{L}W\Omega}_{A/\mathbb{F}_p}$$

which modulo p recover the filtered isomorphisms of Corollary 3.13.

Proof. The proof of this is relatively self-contained so we refer to [8, Thm. 8.15]. □

The proposition implies Theorem 5.9, similarly to the second proof of Theorem 3.1.

6 (Topological) Hochschild and cyclic homologies of $\mathbb{Z}_p$-algebras – a glimpse

It is beyond the scope of this course to discuss in detail the topological Hochschild and cyclic homologies of p-adic algebras, but we nevertheless indicate some of the main results without proof. The goal is essentially to overview the analogous results of §3 and §5 in the case of p-adic algebras.

Remark 6.1 (p-completeness convention)**.** When the input ring A is p-adically complete (but not killed by a power of p), then $HH(A/\mathbb{Z}_p)$, $THH(A)$, etc., as we have already defined them, will contain large amounts of undesirable, junk data. (For example, if S is any perfect $\mathbb{F}_p$-algebra, then we would like $HH(W(S)/\mathbb{Z}_p)$ to be supported in degree 0, since $\mathbb{L}_{W(S)/\mathbb{F}_p}$ should vanish; but these statements are only true after p-completing the complexes.) In this section we therefore adopt the following convention: if A is a p-adically complete ring, then we write $HH(A/\mathbb{Z}_p)$, $THH(A)$, etc. to mean $HH(A/\mathbb{Z}_p)^{\widehat{}}_p$, $THH(A)^{\widehat{}}_p$, where the hat denotes derived p-adic completion.

6.1 The case of perfectoid rings and consequences

We begin with the analogue of Bökstedt's calculation (Theorem 5.1) and the consequences for topological periodic and topological negative cyclic homologies (analogue of Proposition 5.2). Although we stated these only for $\mathbb{F}_p$ we remarked that they held for arbitrary perfect $\mathbb{F}_p$-algebras; we therefore begin by recalling the p-adic analogue of a perfect $\mathbb{F}_p$-algebra:

Definition 6.2. A ring A is *perfectoid* if it satisfies the following conditions:

(i) it is p-adically complete and separated;

(ii) A/pA is semiperfect;

(iii) there exist $\pi \in A$ and $u \in A^\times$ such that $\pi^p = pu$;

(iv) the kernel of Fontaine's map $\theta_A : A_{\mathrm{inf}}(A) \to A$ is principal.

It is not our intention here to review the theory of perfectoid rings, in particular do not discuss Fontaine's ring $A_{\mathrm{inf}}(A) := W(A^\flat)$ or the associated map $\theta_A : A_{\mathrm{inf}}(A) \to A$. We refer to [7] or to [Lecture IV of Bhatt's notes on prisms, available on his webpage].

Example 6.3. (i) An $\mathbb{F}_p$-algebra is perfect if and only if it is perfectoid.

(ii) Let C be a perfectoid field (e.g., an algebraically closed field which is complete under a rank one valuation) of mixed characteristic containing all p-power roots of unity, and let $\mathcal{O}_C \subseteq C$ denote its ring of integers. Then $\mathcal{O}_C$ is a perfectoid ring, which often serves as the base ring in the theory.

Theorem 6.4. *Let A be a perfectoid ring. Then the homotopy groups of $THH(A)$ are given by*

$$THH_n(A) \cong \begin{cases} A & n \text{ even } \geq 0 \\ 0 & n \text{ otherwise} \end{cases}$$

Regarding multiplicative structure, $THH_*(A) = A[u]$ (polynomial on a single variable u), with u any generator of the invertible A-module $THH_n(A) = HH_2(A/\mathbb{Z}_p) = \operatorname{Ker}\theta_A/(\operatorname{Ker}\theta_A)^2$.

Sketch of proof. This is the analogue of Bökstedt's theorem. It is proved by performing various base changes to capture all the data either from characteristic p (i.e., Bökstedt) or rationally (i.e., usual Hochschild homology). See [8, Thm. 6.1]. □

Let $\xi \in A_{\mathrm{inf}}(A)$ be a generator of $\operatorname{Ker}\theta_A$.

Theorem 6.5. *The homotopy groups of $TP(A)$ are given by*

$$TP_n(A) \cong \begin{cases} A_{\inf}(A) & n \text{ even } \geq 0 \\ 0 & n \text{ otherwise} \end{cases}$$

Regarding multiplicative structure, $TP_*(A) = A_{\inf}[\sigma^{\pm 1}]$, where σ is any generator of the invertible $A_{\inf}(A)$-module $TP_2(A)$. The map of graded rings $TC^-_*(A) \to TP_*(A)$ is injective, with image

$$\operatorname{Im}(TC^-_{2n}(A) \to TP_{2n}(A) = \begin{cases} \xi^n TP_{2n}(A) & n \geq 0 \\ TP_{2n}(A) & n \leq 0 \end{cases},$$

i.e., $TC^-_*(A) = A_{\inf}(A)[\xi\sigma, \sigma^{-1}]$, and the canonical morphism $TC^-(A) \to THH(A)$ sends $\xi\sigma$ to u (equivalently, without making any choices, $TC^-_*(A)/\xi \overset{\simeq}{\to} THH_*(A)$ for $* \geq 0$).

Proof. This is proved similarly to Proposition 5.2, noting that the universal property of Fontaine's ring $A_{\inf}(A)$ provides a map $A_{\inf}(A) \to TP_0(A)$ which serves as a replacement for the map $\mathbb{Z}_p \to TP_0(\mathbb{F}_p)$ which we used in the earlier proof. □

The previous theorems lead to analogues of results 5.5–5.8; see [8, §6.3].

6.2 Quasiregular semiperfectoids

The mixed characterisation of a quasiregular semiperfect $\mathbb{F}_p$-algebra is as follows:

Definition 6.6. A ring A is called *quasiregular semiperfectoid* if it satisfies the following conditions:

(i) it is p-adically complete and A/pA is semiperfect;

(ii) it has bounded p^∞-torsion;

(iii) there exists a perfectoid ring S and a map $S \to A$ such that the p-completion of $\mathbb{L}_{A/S}$ is supported in homological degree 1 where it is given by a module $N := H_1(\widehat{\mathbb{L}}_{A/S})$ such that N/p^r is a flat A/p^r-module for all $r \geq 0$.

When A is qrsp in this sense, we remark that the condition on the cotangent complex is then satisfied for *any* map from a perfectoid ring to A (which may in fact be chosen to be surjective).

Example 6.7. (i) Any quasiregular semiperfect $\mathbb{F}_p$-algebra is of course quasiregular semiperfectoid. Nevertheless, we obtain some new presentations: for example, the quasiregular semiperfect $\mathbb{F}_p$-algebra $\mathcal{O}_C^\flat/p^\flat$ is the same as $\mathcal{O}_C/p$.

(ii) The prototypical example of a quasiregular semiperfectoid not of characteristic p is $\mathcal{O}_C\langle T^{\pm 1/p^\infty}\rangle/(T-1)$.

Arguing as in Lemmas 3.8 and 5.10, one again sees that TP_0 of a quasiregular semiperfectoid ring admits a filtration coming from negative cyclic homology:

Lemma 6.8. *Let A be a quasiregular semiperfectoid ring, fix an integral perfectoid ring S and map $S \to A$, and set $N := \pi_1(\widehat{\mathbb{L}}_{A/S})$. Then $THH_*(A)$, $TC^-_*(A)$, $TP_*(A)$, and $\pi_* THH(A)_{hS^1}$ are all supported in even degrees. Moreover, $THH_{2i}(A)$ admits a finite increasing filtration with graded pieces given by the p-completions of $A, N, \Gamma^2_A(N), \dots, \Gamma^i_A(N)$, and $TP_0(A/\mathbb{F}_p)$ admits a complete, descending, $\mathbb{N}$-indexed filtration by ideals such that*

$$\mathrm{Fil}^i \cong TC^-_{2i}(A), \qquad TP_0(A)/\mathrm{Fil}^i \cong \pi_{2i-2} THH(A)_{hS^1}, \qquad \mathrm{gr}^i \cong THH_{2i}(A) \cong \pi_i(\widehat{\mathbb{L}}^i_{A/S})$$

for all $i \geq 0$.

In the case of classical periodic cyclic homology we saw two descriptions of HP_0 of qusasiregular semiperfect rings: either in terms of derived de Rham cohomology or an explicit divided power envelope. We had two similar descriptions of TP_0, either in terms of derived crystalline cohomology or again via a divided power envelope construction. We would like a similar description of TP_0 of any quasiregular semiperfectoid:

Theorem 6.9. *Let A be a qrsp $\mathcal{O}_C$-algebra. Then there are natural isomorphisms of filtered rings*

$$TP_0(A) \cong \widehat{\mathbf{\Delta}}_{A/A_{\mathrm{inf}}} \cong \widehat{A\Omega}_A$$

(The middle and right filtered rings are defined below.)

Proof. This is proved by a rather intricate set of arguments bouncing back and forth between qrsp and formally smooth algebras, and also requires Bhatt–Scholze's forthcoming work on prismatic cohomology. We aren't going to say anything about it. □

6.2.1 Derived $A\Omega$ cohomology

Let R be the p-adic completion of a smooth $\mathcal{O}_C$-algebra; assume that R is *small* in the sense that there exists a formally étale morphism $\mathcal{O}\langle T_1^{\pm 1}, \ldots, T_d^{\pm 1}\rangle \to R$ for some $d \geq 0$ (but do not fix any such morphism). In [7] the authors introduced an $\mathbb{E}_\infty$-A_{inf}-algebra $A\Omega_R \in D(A_{\mathrm{inf}})$ which may be described in any of the following ways:

(i) (Faltings-style Galois cohomology) Assuming that $\operatorname{Spf} R$ is connected, let $\overline{R[\frac{1}{p}]}$ be a "universal cover" of R, i.e., the filtered union of all finite étale extensions of $R[\frac{1}{p}]$ inside a fixed algebraic closure of $\operatorname{Frac} R$. In particular, the geometric Galois group $\Delta := \pi_1^{\text{ét}}(\operatorname{Spec} R)$ acts naturally on $\overline{R[\frac{1}{p}]}$. Next let $\overline{R}$ be the p-adic completion of the integral closure of R inside $\overline{R[\frac{1}{p}]}$ and observe that $\overline{R}$ inherits a continuous action by Δ. It turns out (here the smallness assumption is used) that $\overline{R}$ is integral perfectoid, so we may form its associated Fontaine ring $A_{\mathrm{inf}}(\overline{R})$, which is again equipped with a continuous action by Δ, and the resulting Galois cohomology $R\Gamma_{\mathrm{cont}}(\Delta, A_{\mathrm{inf}}(\overline{R}))$. The first definition of $A\Omega_R$ is
$$A\Omega_R := L\eta_\mu R\Gamma_{\mathrm{cont}}(\Delta, A_{\mathrm{inf}}(\overline{R})),$$
where $L\eta_\mu$ is the décalge functor from [7].

(ii) (q-de Rham complex) The previous definition of $A\Omega_R$ is a priori uncomputable since the geometric Galois group Δ is inaccessible. However, the almost purity theorem and a curious property of the décalage functor overcome this difficulty as follows. Fix a formally étale morphism $\mathcal{O}\langle T_1^{\pm 1}, \ldots, T_d^{\pm 1}\rangle \to R$ and define R_∞ to be the p-adic completion of $R \otimes_{\mathcal{O}\langle \underline{T}^{\pm 1}\rangle} \mathcal{O}\langle \underline{T}^{\pm 1/p^\infty}\rangle$. Then $R_\infty \subseteq \overline{R}$, and the Δ-action on $\overline{R}$ restrictions to R_∞: in fact, on R_∞ the action factors through $\Delta \to \Gamma := \mathbb{Z}_p(1)^d$, where $\mathbb{Z}_p(1)^d$ acts on R_∞ by rescaling the p-power roots of the variables by suitable p-power roots of unity. A consequence of the almost purity theorem for perfectoid rings is that the induced morphism $R\Gamma_{\mathrm{cont}}(\Gamma, A_{\mathrm{inf}}(R_\infty)) \to R\Gamma_{\mathrm{cont}}(\Delta, A_{\mathrm{inf}}(\overline{R}))$ is in fact an almost equivalence; moreover, the almost difference remarkably disappears after applying the décalage functor, i.e.,
$$L\eta_\mu R\Gamma_{\mathrm{cont}}(\Gamma, A_{\mathrm{inf}}(R_\infty)) \xrightarrow{\sim} L\eta_\mu R\Gamma_{\mathrm{cont}}(\Delta, A_{\mathrm{inf}}(\overline{R})) = A\Omega_R$$
But the left side is defined in terms of an explicit action of a pro-cyclic group and can be computed in terms of Koszul complexes; it turns out to be a "q-de Rham complex", for which we refer to [7] for further details.

(iii) (pro-étale cohomology) The problem with both of the previous definitions is that they do not enjoy strong functoriality properties: the q-de Rham description is completely non-functorial as it depends on the chosen coordinates, while the Galois cohomology descriptions depends at least on the choice of an algebraic closure of $\operatorname{Frac} R$. The solution to this problem adopted in [7] is to replace the Galois cohomology $R\Gamma_{\mathrm{cont}}(\Delta, A_{\mathrm{inf}}(\overline{R}))$ by the site-theoretic pro-étale cohomology $R\Gamma_{\text{proét}}(\operatorname{Spa}(R[\frac{1}{p}], R), \mathbb{A}_{\mathrm{inf}})$ of a period sheaf $\mathbb{A}_{\mathrm{inf}}$ on the adic spectrum $\operatorname{Spa}(R[\frac{1}{p}], R)$. Similarly to (ii), these two cohomologies are almost equivalent and become equivalent after décalge, i.e., $A\Omega_R \simeq R\Gamma_{\text{proét}}(\operatorname{Spa}(R[\frac{1}{p}], R), \mathbb{A}_{\mathrm{inf}})$. This yields a definition of $A\Omega_R$ which is functorial and can be easily extended to arbitrary smooth, formal $\mathcal{O}$-schemes.

Definition 6.10. For an arbitrary p-complete $\mathcal{O}_C$-algebra A, let $A\Omega_A$ denote the (p, ξ)-adic completion of the left Kan extension of $A\Omega_-$ from p-adic completions of smooth $\mathcal{O}$-algebras.

When R is a formally smooth $\mathcal{O}_C$-algebra, then $A\Omega_R$ is moreover equipped with a complete filtration whose graded pieces are given by truncations of the de Rham complex of R. By left Kan extending this filtration and mimicking the arguments of §5.2.1, one checks that if A is a qrsp $\mathcal{O}_C$-algebra then $A\Omega_A$ is a supported in degree 0, where it is given by a complete filtered ring whose graded pieces are related to derived de Rham cohomology; in that case we write $\widehat{A\Omega_A}$ for the completion of A with respect to this filtration.

6.2.2 Prisms and δ-rings (following Bhatt–Scholze)

Let A be a qrsp $\mathcal{O}_C$-algebra. The new theory of prismatic cohomology, in which (very roughly) divided powers are replaced by p-derivations, associates to A its prismatic cohomology $\mathbf{\Delta}_{A/A_{\mathrm{inf}}}$. Since A is qrsp this cohomology is supported in degree zero, where it is given by a sort of period ring constructed in terms of p-derivations. We now explain this construction.

Fix an integral perfectoid $\mathcal{O}_C$-algebra S equipped with a surjection $S \to A$. Let $A_{\mathrm{inf}}(S)[\frac{1}{\xi}]^\frown$ be the p-adic completion of $A_{\mathrm{inf}}(S)[\frac{1}{\xi}]$ and observe that the usual Witt vector Frobenius φ on $A_{\mathrm{inf}}(S) = W(S^\flat)$ extends uniquely to a Frobenius lift on this completion (indeed, one just needs to check that $\varphi(\xi)$ is a unit mod p in $A_{\mathrm{inf}}(S)[\frac{1}{\xi}]$, but this is clear since $\varphi(\xi) \equiv \xi^p \bmod p$). We denote by $\delta : A_{\mathrm{inf}}(S)[\frac{1}{\xi}]^\frown \to A_{\mathrm{inf}}(S)[\frac{1}{\xi}]^\frown$ the associated p-derivation, i.e., $\delta(f) := \frac{1}{p}(\varphi(f) - f^p)$.

Let $\mathbf{\Delta}_{A/A_{\mathrm{inf}}}$ be the (p,ξ)-adic completion of the $A_{\mathrm{inf}}(S)$-subalgebra of $A_{\mathrm{inf}}(S)[\frac{1}{\xi}]^\frown$ generated by terms of the form $\delta^n(\frac{f}{\xi})$ for all $f \in \mathrm{Ker}(A_{\mathrm{inf}}(S) \xrightarrow{\theta} S \to A)$ and $n \geq 0$. We equip $\mathbf{\Delta}_{A/A_{\mathrm{inf}}}$ with a Nygaard-style filtration $\mathcal{N}^{\geq i}\mathbf{\Delta}_{A/A_{\mathrm{inf}}} := \{g \in \mathbf{\Delta}_{A/A_{\mathrm{inf}}} : \varphi(g) \in \xi^i \mathbf{\Delta}_{A/S}\}$ for $i \geq 0$, and let $\widehat{\mathbf{\Delta}}_{A/A_{\mathrm{inf}}}$ be its completion for this filtration.

6.3 Smooth algebras

It remains only to say something about the topological cyclic homologies of a smooth $\mathcal{O}_C$-algebra; for these we have the following analogue of Theorems 3.1 and 5.9:

Theorem 6.11. *Let R be a p-adic formally smooth $\mathcal{O}_C$-algebra. Then $TP(R)$ and $TC^-(R)$ admit natural, complete, descending $\mathbb{Z}$-indexed filtrations whose n^{th} graded pieces are given respectively by*

$$\mathcal{N}^{\geq n} A\Omega_R, [2n] \qquad A\Omega_R[2n],$$

where we use the $A\Omega_R$ complex and its filtration from §6.2.1.

Sketch of proof. Similar to the proof of the aforementioned theorems: in a p-adic version of the quasisyntomic topology, R may be covered by quasiregular semiperfectoid $\mathcal{O}_C$-algebras. On the TP and TC^- of these latter algebras we impose the two-speed Postnikov filtration, whose graded pieces are calculated by Theorem 6.9. Descending back down to $TP(R)$ and $TC^-(R)$ then implies the theorem. □

Projects

A Motivic cohomology with modulus in characteristic p
(Themes: derived de Rham–Witt complexes)

A.1 Motivation: classical motivic cohomology

To any finite type scheme X over a field k, Bloch associated his *cycle complex*

$$z^n(X) = [z^n(X,0) \leftarrow z^n(X,1) \leftarrow \cdots]$$

This is constructed more naturally as a simplicial abelian group, in which $z^n(X,i)$ consists of formal sums of irreducible, codimension-n subvarieties of $X \times_k \Delta^i_k$ (where Δ^i_k is the spectrum of $k[t_1,\dots,t_i]/(t_1 + \cdots + t_i - 1)$) which intersect all faces correctly. The homology of this chain complex is Bloch's higher Chow groups $CH^n(X,i) := H_i(z^n(X))$; up to reindexing these are the same as Voevodsky's motivic cohomology $H^i_{\mathcal{M}}(X,\mathbb{Z}(q)) := CH^q(X, 2q-p) = H^p(z^q(X)[-2q])$.

We recall also that each presheaf $X \supseteq U \mapsto z^n(U,i)$ turns out to be a Zariski sheaf, and that this complex of Zariski sheaves $U \mapsto z^n(U)$ even satisfies Zariski descent; therefore the above definitions can be rephrased in terms of the hypercohomology of this complex of sheaves. In particular, introducing the complex of Zariski sheaves $\mathbb{Z}(i)_X : U \mapsto z^n(U)[-2n]$, we have $H^i_{\mathcal{M}}(X,\mathbb{Z}(q)) = \mathbb{H}^i_{\mathrm{Zar}}(Z,\mathbb{Z}(q)_X)$. (In an attempt to avoid mistakes when shifting and passing between (c)ohomological indexing, it might be helpful to note the following: $\mathbb{Z}(q)_X$ cohomologically vanishes in degrees $> 2q$, since each $z^q(U)$ is homologically supported in degrees ≥ 0; also, $\mathbb{Z}(q)_X$ should be cohomologically supported in degrees ≥ 0, but this is precisely the Beilinson–Soulé vanishing conjecture.)

Here is the statement we really care about:

Theorem A.1 (Bloch–Lichtenbaum, Levine, Voevodsky)**.** *Assuming X is smooth over k (otherwise one should replace $K(X)$ by $K'(X)$ in what follows), then the K-theory spectrum $K(X)$ admits a complete, descending $\mathbb{N}$-indexed filtration whose i^{th} graded pieces is*

$$R\Gamma_{\mathcal{M}}(X,\mathbb{Z}(i))[2i] := \mathbb{H}_{\mathrm{Zar}}(Z,\mathbb{Z}(q)_X)[2q] = z^q(X)$$

Therefore there is an associated "Atiyah–Hirzebruch" spectral sequence converging to the K-groups $K_*(X)$.

A.2 Motivic cohomology beyond the smooth case

An open problem is to find an analogous motivic filtration on the K-theory of arbitrary schemes (and to relate it to existing objects, such as algebraic cycles). The particular case which has attracted most attention in recent years had aimed at constructing such a motivic filtration on the relative K-group $K(X,D)$, where X is smooth and $D \hookrightarrow X$ is a simple normal crossing divisor (which is not necessarily reduced), such that the graded pieces are related to so-called "Chow groups with modulus".

Let X be a smooth variety over a perfect field k of characteristic p, and let $D = \bigcup_{i=1}^c D_i \hookrightarrow X$ be a reduced normal crossing divisor. For a tuple of positive integers $\underline{r} = (r_1,\dots,r_c)$, we write $\underline{r}D := \cup_{i=1}^c r_i D_i$, where $r_i D_i$ means the r_i-infinitesimal thickening of D. In other words, if D is defined locally by the equation $t_1 \cdots t_c = 0$, then $\underline{r}D$ is defined by $t_1^{r_1} \cdots t_c^{r_c} = 0$.

As explained above, the relative K-theory $K(X,\underline{r}D)$ should be equipped with a motivic filtration whose graded pieces are related to higher Chow groups with modulus, which themselves are known to be close to de Rham–Witt groups and similar objects.

The K-theory $K(X,\underline{r}D)$ is built from $K(X,D)$ and $K(\underline{r}D,D)$, which one may refer to respectively as the *tame* and *wild* parts. Since $D \hookrightarrow \underline{r}D$ is an infinitesimal thickening in characteristic p, the trace map induces an equivalence $K(\underline{r}D,D) \simeq TC(\underline{r}D,D)$; the latter is built from $TC(\underline{r}D)$ and $TC(D)$, which carry filtrations from [8] with i^{th} graded piece given by the Zariski hypercohomology of

$$\operatorname{hofib}\left(\mathcal{N}^{\geq i}\mathbb{L}W\Omega_{\underline{r}D/k} \xrightarrow{\frac{\varphi}{p^i}-1} \mathbb{L}W\Omega_{\underline{r}D/k}\right)$$

(resp. replacing $\underline{r}D$ by D), where $\mathbb{L}W\Omega_{-/k}$ is the derived de Rham–Witt complex equipped with its Nygaard filtration.

Question A.2. Can this be used to build the desired motivic filtration on $K(X, \underline{r}D)$? In other words:

(i) Firstly, can the above Frobenius fixed points be related to other objects appearing in the theory of motivic cohomology with modulus? How does the derived de Rham-Witt complex of a simple normal crossing divisor look? Is Bhatt–Lurie–Mathew's saturated de Rham–Witt complex better adapted?

(ii) Secondly, once we have the right filtration on the wild part, can it be glued to a filtration on the tame part to get the desired filtration on $K(X, \underline{r}D)$?

Question A.3. A closely related but more immediately accessible problem is simply to compute the *THH*, etc. (including K-theory) of rings B such as $k[t]/t^r$ and $k[t_1, \dots, t_c]/(t_1^{r_1}, \dots, t_c^{r_c})$ (warning: the latter ring is not the one defining the ncd $\underline{r}D$ above). Many such results are already available (see particularly Hesselholt–Madsen and Angeltveit–Gerhardt–Hill–Lindenstrauss respectively), but it would be nice to have a more natural presentation in terms of (derived) de Rham–Witt theory. The point is that the relative K-groups can be computed in terms of the topological cyclic homology $TC(B)$, which fits into a fibre sequence

$$TC(B) \longrightarrow TC^{-}(B) \xrightarrow{\varphi - 1} TP(B),$$

all of which carry filtrations similar to the discussion above.

More ambitiously, what about rings like $\mathbb{Z}/p^r$ or $\mathcal{O}/p^r$, where $\mathcal{O}$ is the ring of integers of a perfectoid field of characteristic zero? (This seems hard without the forthcoming theory of prismatic cohomology of Bhatt–Scholze, which will provide an interpretation of the objects to be calculated in terms of p-derivations.)

B Quasisyntomic coefficients
(Themes: quasisyntomic site, crystals)

Let R be a smooth algebra over a perfect field of characteristic p (more generally one could allow R to be any quasisyntomic $\mathbb{F}_p$-algebra, or replace $\operatorname{Spec} R$ by a non-affine variety).

On the quasisyntomic site we have a sheaf TP_0^{sh} which associates to each quasiregular semiperfect A the (filtered) ring

$$TP_0(A) \cong \widehat{\mathbb{A}}_{\mathrm{crys}}(A) \cong \widehat{\mathbb{L}W\Omega}_{A/\mathbb{F}_p}.$$

Question B.1. How do locally free sheaves of TP_0^{sh}-modules on the quasisyntomic site over R look? Are they related to crystals on the crystalline site of R? It might be helpful to first consider the mod p-situation $HP_0^{\mathrm{sh}} = TP_0^{\mathrm{sh}}/p = \widehat{\mathbb{L}\Omega}_{-/\mathbb{F}_p}$. It might also be better to study sheaves of filtered modules, or perhaps modules with a Frobenius.

How are such objects related to modules of cyclotomic spectra over $THH(R)$?

In theory the analogue of the previous question should already have been answered if R is a smooth algebra over a field k of characteristic 0: then one expects modules with S^1-action over $HH(R/k)$ to be related to R-modules equipped with a filtration and connection satisfying Griffiths transversality. But I do not know of a reference.

C Smoothness conditions and Cartier isomorphisms
(Theme: cotangent complex)

C.1 Smoothness conditions

There are three possible nilpotence condition which one can impose on an ideal I:

- *nilpotent* means that there exists $n \geq 1$ such that $I^n = 0$;
- *nil* means that there exists $n \geq 1$ such that $x^n = 0$ for all $x \in I$.
- *locally nilpotent* means that for each $x \in I$ there exists $n \geq 1$ such that $x^n = 0$.

Obviously nilpotent $\Rightarrow$ nil $\Rightarrow$ locally nilpotent.

A ring homomorphism $A \to B$ is called *formally smooth* if for each A-algebra C and nilpotent ideal $I \subseteq C$, any A-algebra homomorphism $B \to C/I$ lifts to an A-algebra homomorphism $B \to C$; when the lifting is moreover always unique, one says that $A \to B$ is *formally étale*. By replacing "nilpotent" in this definition by "nil" or "locally nilpotent", we may similarly define *n-formally smooth/étale* and *ln-formally smooth/étale*. (This terminology is totally non-standard; the concept of n-formally smooth appears elsewhere in the literature, notably in the theory of crystalline cohomology and especially in the thesis of Berthelot, where it is called "quasi-smooth".)

One can also ask whether B is a filtered colimit of smooth étale A-algebras, in which case we say it is *ind-smooth*; one defines *ind-étale similarly.* In summary we arrive at the following implications:

ind-smooth $\Rightarrow$ ln-formally smooth $\Rightarrow$ n-formally smooth $\Rightarrow$ formally smooth

and similarly for étale.

There is in fact also a class which is between ind-étale and and ln-formally étale (and strictly larger than ind-étale), namely weakly étale. Recall here Bhatt–Scholze's notion of *weakly étale*, which means that both $A \to B$ and $\mu : B \otimes_A B \to B$ are flat. According to [9, Thm. 1.3], if $A \to B$ is weakly étale then there exists a faithfully flat map $B \to C$ such that $A \to B$ is ind-étale (i.e., weakly étale maps are locally ind-étale in the flat topology), whence flat descent implies that $A \to B$ is ln-formally étale. It seems difficult to construct an ln-formally étale map which is not weakly eétale, and so Nikolaus has raised the following question:

Question C.1. Is ln-formally étale actually equivalent to weakly étale? Consider first the case of an extension of fields $k \to k'$ (in which case weakly étale is equivalent to ind-étale, i.e., that k' is an algebraic separable extension of k).

We can also consider conditions coming from the cotangent complex: say that $A \to B$ is *L-smooth* if $\mathbb{L}_{B/A}$ is supported in degree 0 and $\Omega^1_{B/A}$ is a projective B-module; say it is *L-étale* if $\mathbb{L}_{B/A} \simeq 0$. It is known that

L-smooth $\Rightarrow$ formally śmooth

whence (since formally étale is equivalent to formally smooth and vanishing of Ω^1) we get

L-étale $\Rightarrow$ formally étale

but the converse is false by the next example:

Example C.2. See [Stacks project, Lem. 102.37.1] for an example of a formally étale morphism with non-vanishing cotangent complex.

Gabber has given an example of a non-reduced (hence not ind-smooth) $\mathbb{F}_p$-algebra B such that $\mathbb{L}_{B/\mathbb{F}_p} \simeq 0$, i.e., L-étale does not imply ind-smooth. You can find the construction of B on Bhatt's webpage. According to the remark between 1.9 and 1.10 of [12], Gabber claims that the construction can be modified (by replacing the perfections of the B_i which occur in Bhatt's write-up by the extensions obtained by adjoining all squares of elements) to produce a $\mathbb{Q}$-algebra with similar properties (see the next question).

Question C.3. (i) Carry out Gabber's suggestion to carefully construct a $\mathbb{Q}$-algebra B which is quasismooth (see below) but not ind-smooth.

(ii) Bhatt asks the following: Does there exist an L-étale $\mathbb{Q}$-algebra which is not ind-étale?

(iii) Remaining on the subject of special base rings, is every formally étale $\mathbb{F}_p$-algebra automatically L-étale?

(iv) Does L-smooth imply ln-formally smooth? Similarly for étale?

Recall from Remark 3.18 that we say $A \to B$ is *quasismooth* if $\mathbb{L}_{B/A}$ is supported in degree 0 and $\Omega^1_{B/A}$ is a flat B-module. So

L-smooth $\Rightarrow$ quasismooth

It turns out that if A and B are Noetherian then $A \to B$ is quasismooth if and only if it is ind-smooth, and so all our smoothness conditions are then equivalent. This is a consequence of Néron–Popescu desingularisation and a hard theorem of M. André [1, Thm. 30, pg. 331]. André's theorem is actually more general (for example, he can say something even if A is not Noetherian, as long as $A \to B$ is flat), and it might be useful.

C.2 Incorporating the Cartier isomorphism

Let A be an $\mathbb{F}_p$-algebra. Then the *inverse Cartier map*

$$C^{-1} : \Omega^n_{A/\mathbb{F}_p} \to H^n(\Omega^\bullet_{A/\mathbb{F}_p})$$

is the linear map characterised by

$$C^{-1}(f dg_1 \wedge \cdots \wedge dg_n) = f^p g_1^{p-1} \cdots g_n^{p-1} dg_1 \wedge \cdots \wedge dg_n$$

for all $f, g_1, \ldots, g_n \in A$. In [26], we say that A is *Cartier smooth* if the following conditions are satisfied:

(Sm1) $\mathbb{F}_p \to A$ is quasismooth;

(Sm2) the inverse Cartier maps are isomorphisms for all $n \geq 0$.

The above criteria are of course satisfied if A is a smooth $\mathbb{F}_p$-algebra, or more generally if A is a regular Noetherian $\mathbb{F}_p$-algebra by Néron–Popescu desingularisation (an alternative proof avoiding Néron–Popescu may be found in the recent work of Bhatt–Lurie–Mathew [6, Thm. 9.5.1]), or simply an ind-smooth $\mathbb{F}_p$-algebra, or even a smooth algebra over a perfect $\mathbb{F}_p$-algebra. But They are also satisfied if A is a valuation ring of characteristic p: criterion (Sm1) is due to Gabber–Ramero [19, Thm. 6.5.8(ii) & Corol. 6.5.21], while the Cartier isomorphism (Sm2) is a result of Gabber obtained by refining his earlier work with Ramero, the proof of which may be found in the appendix of [27].

Note that condition (Sm2) is not a consequence of condition (Sm1), as explained in [6, Warning 9.6.3]: Bhatt–Gabber's Example C.2 is a semiperfect, non-perfect $\mathbb{F}_p$-algebra B such that $\mathbb{L}_{B/\mathbb{F}_p} \simeq 0$; then the inverse Cartier map in degree $n = 0$ identifies with $\varphi : A \to A$, which has non-zero kernel. Condition (Sm2) is designed to overcome this sort of pathological behaviour: for example, Cartier smooth + L-étale is equivalent to perfect, which implies formally étale (even n-formally étale).

Question C.4. (i) Assuming that A is quasismooth over $\mathbb{F}_p$, observe that condition (Sm2) is equivalent to the adjunction $\mathbb{L}\Omega^\bullet_{A/\mathbb{F}_p} \to \Omega^\bullet_{A/\mathbb{F}_p}$ being an equivalence (c.f., [6, 9.6.2]). Take care of the difference between $\mathbb{L}\Omega^\bullet$ and its Hodge completion $\widehat{\mathbb{L}\Omega^\bullet}$.

(ii) Is there a good notion (say, closed under composition) of what it means for a morphism of $\mathbb{F}_p$-algebras to be Cartier smooth? If so, can you prove that $\mathcal{O}_{C_p}/p \to \mathcal{O}/p$ is Cartier smooth? (For the notation see Project C.3. Maybe one can replace $\mathcal{O}$ by $\overline{\mathcal{O}}$ and then note that $\mathcal{O}_{C_p}/p \to \overline{\mathcal{O}}/p$ can be rewritten as $\mathcal{O}^\flat_{C_p}/p^\flat \to \overline{\mathcal{O}}^\flat/p^\flat$, to which one might be able to apply existing results in characteristic p?)

(iii) How does Cartier smooth compare to Bhatt–Lurie–Mathew's "universal Cartier isomorphism"?

C.3 Extending to mixed characteristic

The relation of Cartier smoothness to the material of §5 is as follows: for any $\mathbb{F}_p$-algebra A, we write $\mathbb{L}W\Omega_{A/\mathbb{F}_p} := \mathrm{Rlim}_r \mathbb{L}W_r\Omega_{A/\mathbb{F}_p}$, or equivalently the p-adic completion of the left Kan extension of $W\Omega_{-/\mathbb{F}_p}$ on A, and we let $\widehat{\mathbb{L}W\Omega}_{A/\mathbb{F}_p}$ be its Nygaard completion; these are connected by natural morphisms

$$\mathbb{L}W\Omega_{A/\mathbb{F}_p} \longrightarrow \widehat{\mathbb{L}W\Omega}_{A/\mathbb{F}_p} \longrightarrow W\Omega_{A/\mathbb{F}_p}$$

The left object is perhaps the most classical, the middle object is the right one to describe $TP(A)$, and the right one is the most computable. The key point is the following: if A is Cartier smooth over $\mathbb{F}_p$ (e.g., a valuation ring of characteristic p) then the morphisms are equivalences (this can be extracted from the arguments of [26]).

The goal of the rest of this project is to establish a similar result in mixed characteristic; some familiarity with [7] is required.

Let C_p be an algebraically closed, non-archimedean field of mixed characteristic, e.g., $\mathbb{C}_p$, and let $\mathcal{O}_{C_p}$ denote its ring of integers. Let C be a complete valued field extension of C_p (possibly highly transcendental and not necessarily algebraically closed) with corresponding ring of integers $\mathcal{O} \supseteq \mathcal{O}_{C_p}$ (if C has rank > 1, then we could even let $\mathcal{O}$ be a non-maximal valuation subring). The project concerns various cohomology theories attached to $\mathcal{O}$. Note that Gabber–Ramero [19, Thm. 6.5.8(ii) & Corol. 6.5.21] again tells us that $\mathcal{O}_C \to \mathcal{O}$ is quasismooth. (In contrast, although $\mathbb{Z}_p \to \mathcal{O}_C$ also has cotangent complex supported in degree 0, the $\mathcal{O}_C$-module $\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}$ is far from being flat: its p-torsion $\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}[p]$ is a rank one $\mathcal{O}_C/p$-module.)

Let $\overline{C}$ be the completed algebraic closure of C, and $\overline{\mathcal{O}} \subseteq \overline{C}$ its ring of integers; let Δ be the absolute Galois group of C. Let $\overline{\mathcal{O}}^\flat$ be the tilt of $\overline{\mathcal{O}}$. The first cohomology we are interested in is the Galois cohomology $R\Gamma(\Delta, \overline{\mathcal{O}})$ and its décalage $\widetilde{\Omega}_{\mathcal{O}} := L\eta_{\zeta_p - 1} R\Gamma(\Delta, \overline{\mathcal{O}})$ (Note: similarly to the smooth case studied in [7], it should be equivalent to define $\widetilde{\Omega}_{\mathcal{O}}$ as $L\eta_{\zeta_p-1} R\Gamma_{\text{proét}}(X, \widehat{\mathcal{O}}^+_X)$ where $X := \mathrm{Spa}(\mathcal{O}, C)$). The following Hodge–Tate comparison plays the role of the Cartier isomorphism in mixed characteristic:

Question C.5. Do there exist natural (but ignoring Breuil–Kisin–Fargues twists) isomorphisms

$$\Omega^i_{\mathcal{O}/\mathcal{O}_{C_p}} \cong H^i(\widetilde{\Omega}_{\mathcal{O}})$$

for $i \geq 0$? Compare with [7, §8] for the case of a smooth $\mathcal{O}_{C_p}$-algebra in place of $\mathcal{O}$. By mimicking the argument in the smooth case (and taking advantage of the aforementioned result of Gabber–Ramero), it should be possible to construct a comparison map in the direction $\to$. To check that it is an isomorphism, maybe reduce modulo p and relate it to the Cartier map for $\mathcal{O}/p$; I suspect one needs to know that $\mathcal{O}_{C_p}/p \to \mathcal{O}/p$ is Cartier smooth, which is why I asked it in Question C.4.

Update: After some discussions with Bhatt, we have decided that the case of an arbitrary complete valued field extension C is probably too ambitious; one should perhaps assume that the $\mathbb{F}_p$-algebra $\mathcal{O}/p$ admits a finite set of generators over its subring of p^{th}-powers. Lifting these to generators to units $t_1, \ldots, t_d \in \mathcal{O}$ should provide a basis $dt_1, \ldots, dt_d$ of the module $\Omega^1_{\mathcal{O}/\mathcal{O}_C}$; meanwhile, the resulting map $\mathcal{O}\langle T_1^{\pm 1}, \ldots, T_d^{\pm 1}\rangle \to \mathcal{O}$ provides the analogue of the framing which usually appears in p-adic Hodge theory. Without this map it seems hard to attack the question. Note that such valuation rings $\mathcal{O}$ are reasonably abundant: take any smooth $\mathcal{O}_C$-algebra R and define $\mathcal{O}$ to be the p-adic completion of $R_{\mathfrak{m}R}$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_C$.

Next let $A_{\inf}(\overline{\mathcal{O}}) := W(\overline{\mathcal{O}}^\flat)$ be Fontaine's infinitesimal period ring construction of $\mathcal{O}$, and

$$A\Omega_{\mathcal{O}} := L\eta_\mu R\Gamma(\Delta, A_{\inf}(\overline{\mathcal{O}})).$$

Assuming that the previous question can be answered in the affirmative, then I am confident that it can be shown that $A\Omega_{\mathcal{O}}$ is exactly the prismatic cohomology of $\mathcal{O}$, that it is already Nygaard complete, and that the motivic filtration on $TP(\mathcal{O})$ has graded pieces given by $A\Omega_{\mathcal{O}}[2i]$, for $i \in \mathbb{Z}$.

Question C.6. Is there any remotely explicit description of the complexes $\widetilde{\Omega}_{\mathcal{O}}$ or $A\Omega_{\mathcal{O}}$, comparable to the q-de Rham complex in the case of a smooth $\mathcal{O}_{C_p}$-algebra in place of $\mathcal{O}$?

D Computing $THH(\mathbb{Z})$

Theme: spectral sequence calculation

Here are two useful results for analysing the topological Hochschild homology of a ring A.

(i) M. Bökstedt's [15, Thm. 4.1.0.1] calculation of the groups $THH_n(\mathbb{Z})$; they are $= \mathbb{Z}/m\mathbb{Z}$ if $n = 2m \geq 0$, and otherwise they vanish.

(ii) Pirashvili–Waldhausen's [33, Thm. 4.1] first quadrant spectral sequence

$$E^2_{ij} = HH_i(A, THH_j(\mathbb{Z}, A)) \implies THH_{i+j}(A).$$

Here $THH(\mathbb{Z}, A)$ refers to THH of the integers with coefficients in A, whose homotopy groups will be given by

$$THH_n(\mathbb{Z}, A) \cong \begin{cases} A/mA & n = 2m-1 \\ A[m] & n = 2m. \end{cases}$$

Similarly $HH(A, M)$ denotes HH of A with coefficients in the A-module M.

For example, if one combines these two results with Bökstedt's calculation of $THH(\mathbb{F}_p)$, then carefully chasing through the spectral sequence can be used to compute THH_* of the ring of integers $\mathcal{O}$ of a mixed characteristic perfectoid field. However, it is also possible to calculate $THH_*(\mathcal{O})$ without using (i), as was done in [8, Thm. 6.1].

Question D.1. By using the description of $THH_*(\mathcal{O})$, as we vary the residue characteristic of $\mathcal{O}$, can the P-W spectral sequence be run backwards to reprove Bökstedt's description of $THH_*(\mathbb{Z})$?

E A filtration on HH etc. depending on coordinates
(Theme: classical cyclic homology)

Let k be a base ring and $S := k[x_1, \ldots, x_n]$ a polynomial ring over k; let V be the free k-module on $x_1, \ldots, x_n$, so that $S = \mathrm{Sym}_k V$; more generally the following works for any free V-module k, or for S being étale over $\mathrm{Sym}_k V$.

A different proof of the HKR isomorphism of Theorem 2.8 uses an argument in the case of polynomial algebras which is a quasi-isomorphism of mixed complexes

$$\text{mixed complex with zero } B\text{-differential corresponding to } \Omega^{\bullet}_{S/k} \\ \xrightarrow{\sim} (HH(S/k), B)$$

See [29, 3.2.2&3.2.3]. The maps defining this depend on writing $\Omega^n_{S/k} = S \otimes_k \bigwedge^n_k V$, which of course depends on the choice of co-ordinates (more precisely, only on choice of the free module V they span). But from this quasi-isomorphism, we see that the statement of Theorem 2.22 remains true for S, even though we did not assume that $k \supseteq \mathbb{Q}$; the problem is that the decompositions appears a priori to depend on the coordinates.

Question E.1. When k is a perfect field of characteristic p, are the resulting filtrations on $HC^-(S/k)$, $HP(S/k)$, $HC(S/k)$ the same as the filtrations we constructed in Theorem 3.1? Note that if this is true, then there will be two interesting consequences: (1) The filtration arising from the above construction does not depend on choice of coordinates; (2) The filtrations of Theorem 3.1 are split (though the splitting depends on choice of coordinates).

F An elementary description of $HP_0(A/\mathbb{F}_p)$

Let A be a quasiregular semiperfect $\mathbb{F}_p$-algebra. Logically the following question is redundant, but it is nevertheless tantalising:

Question F.1. Is it possible to give a direct proof of the isomorphisms of Proposition 3.13 without using topological periodic cyclic homology or Kan extending from the smooth case?

More precisely, as we explained in the second paragraph of the proof of Proposition 3.13, it would be nice to directly construct a comparison map $D_{A^\flat}(I) \to HP_0(A/\mathbb{F}_p)$. Since the target $HP_0(A/\mathbb{F}_p)$ is an algebra over $HP_0(A^\flat/\mathbb{F}_p) = A^\flat$ (to prove this equality, use the vanishing of $\mathbb{L}_{A^\flat/\mathbb{F}_p}$ to see that $HP(A^\flat/\mathbb{F}_p) \stackrel{\simeq}{\to} HP(A^\flat/A^\flat)$), it remains to show that $HP_0(A/\mathbb{F}_p)$ admits divided powers along its ideal $\mathrm{Fil}^1 = \mathrm{Ker}(HP_0(A/\mathbb{F}_p) \to A)$ (where we use the filtration of Lemma 3.8).

By using some formal tricks (as in [8, Props. 8.12 & 8.15]) one can try to reduce the existence of divided powers to the universal case $A = \mathbb{F}_p[t^{1/p^\infty}]/t-1$; this ring is the group algebra of $G := \mathbb{Q}_p/\mathbb{Z}_p$ over $\mathbb{F}_p$. Since G is p-divisible, a standard group homology calculation (e.g., Brown, Homology of groups §V.6) identifies the graded $\mathbb{F}_p$-algebra $H_*(G, \mathbb{F}_p)$ with the divided power algebra $\Gamma^*_{\mathbb{F}_p}(V)$ on the p-torsion $V := G[p] = H_2(G, \mathbb{F}_p)$. Proposition 2.24 therefore identifies $HP_0(G/\mathbb{F}_p)$ with $\widehat{\Gamma}^*_{\mathbb{F}_p}(V)$, the completion of $\Gamma^*_{\mathbb{F}_p}(V)$ with respect to the divided power filtration. In conclusion, the natural map $HP_0(G/\mathbb{F}_p) \to HP_0(\mathbb{F}_p[G]/\mathbb{F}_p) = HP_0(A/\mathbb{F}_p)$ shows that the target does contain some divided powers.

One of the problems is showing that this production of divided powers in $HP_0(A/\mathbb{F}_p)$ is independent of the chosen identification $A = \mathbb{F}_p[t^{1/p^\infty}]/t-1$. To do this it might be helpful to instead work with the surjection $\mathbb{F}_p[A^\times] \to A$, having reduced to the case that A is local and its units are p-divisible. Or try to lift to the p-torsion-free case by using Kaledin's Hochschild–Witt construction?

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