A k-alg

$(D(k), \otimes_k) \ni HH(A|k) = HH(A|\mathbb{Z}) \underset{HH(k/\mathbb{Z})}{\otimes} k$

$\cap|$

$(D(\mathbb{Z}), \otimes_{\mathbb{Z}}) \ni HH(A/\mathbb{Z})$

richer

$\cap H$

$(\ ?\ , \otimes_?\ )$

$A \underset{k}{\otimes} \cdots \underset{k}{\otimes} A$

$||$

$(A \underset{\mathbb{Z}}{\otimes} \cdots \underset{\mathbb{Z}}{\otimes} A) \underset{(k \underset{\mathbb{Z}}{\otimes} \cdots \underset{\mathbb{Z}}{\otimes} k)}{\otimes} k$

- symmetric monoidal $\otimes$
- limits & colimits?
- .....
- dg cat. or simplicial | ∞-cat

No! Can't do better than $D(\mathbb{Z})$!

Yes – best is Sp = Spectra $= D(\mathbb{S})$

- $\mathbb{F}_p$ $\quad HP_0(\mathbb{F}_p | \mathbb{F}_p) = \mathbb{F}_p$

$HP_0(\mathbb{F}_p / \mathbb{Z}) = \mathbb{Z}_p \oplus$ junk

Spectra will kill this.

<u>Today</u>: THH etc. of $\mathbb{F}_p$-algs
$$\overset{\shortparallel}{\mathrm{HH}}(-/\mathbb{S})$$

<u>Thm</u> (Bökstedt): $\mathrm{THH}_{\mathrm{odd}}(\mathbb{F}_p) = 0$ and

$$\mathrm{THH}_{2*}(\mathbb{F}_p) = \mathbb{F}_p[u]$$

where $u \in \mathrm{THH}_2(\mathbb{F}_p)$.

<u>Consequence</u>: For any $\mathbb{F}_p$-alg A, get

$$\mathrm{THH}(A)[2] \xrightarrow{u} \mathrm{THH}(A) \longrightarrow \mathrm{HH}(A/\mathbb{F}_p)$$

$$\mathrm{HH}(A/\mathbb{F}_p) = \mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathbb{F}_p)} \mathbb{F}_p \overset{\text{Bökstedt}}{=} \mathrm{THH}(A)/u$$

take $\Downarrow$ homotopy

$$\mathrm{THH}_0(A) \xrightarrow{\cong} \mathrm{HH}_0(A/\mathbb{F}_p) = A$$

$$\mathrm{THH}_1(A) \xrightarrow{\cong} \mathrm{HH}_1(A/\mathbb{F}_p) = \Omega^1_{A/\mathbb{F}_p}$$

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & THH_1 \arrow[r] \arrow[d, phantom, "\parallel"] & THH_3 \arrow[r] & HH_3 \arrow[r] & THH_0(A) \arrow[r] \arrow[d, phantom, "\parallel"] & THH_2(A) \arrow[r] & HH_2(A/\mathbb{F}_p) \arrow[r] & \cdots \\
& \Omega^1 & & & A & & &
\end{tikzcd}$$

<u>Thm</u> (Hesselholt's HKR thm): If A is a smooth $\mathbb{F}_p$-alg, then

$$THH_*(A) \cong \Omega^*_{A/\mathbb{F}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_p[u]$$

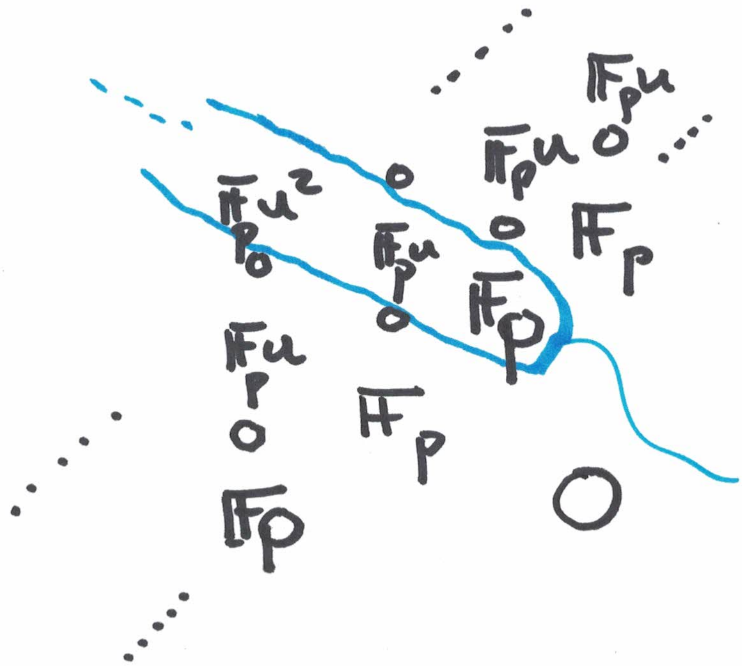
ie) $$THH_n(A) \cong \bigoplus_{i=0}^{n/2} \Omega^{n-2i}_{A/\mathbb{F}_p}$$

<u>Proof</u>:

$$\begin{tikzcd}
0 \arrow[r] & THH_{n-2}(A) \arrow[r] & THH_n(A) \arrow[r] & HH_n(A/\mathbb{F}_p) \arrow[r] & 0 \\
& & & \Omega^n_{A/\mathbb{F}_p} \arrow[u, "\cong"', "\text{Classical HKR}"] \arrow[ul, dashed] &
\end{tikzcd}$$

$\square$

$\underline{TP(\mathbb{F}_p)} = THH(\mathbb{F}_p)^{tS^1} = HP(\mathbb{F}_p/\mathbb{S})$



$\Rightarrow TP_*(\mathbb{F}_p)$

$\to TP_0(\mathbb{F}_p)$

Formal : $TP_0(\mathbb{F}_p)$ complete filtered rings with associated graded $\mathbb{F}_p[u]$.

2 possibilities : $\mathbb{F}_p[\![u]\!]$

$\mathbb{Z}_p \leftarrow$ this is what we get

Thm $\|$

$TP_0(\mathbb{F}_p)$

<u>Thm</u> : $TP_*(\mathbb{F}_p) \cong \mathbb{Z}_p[\sigma^{\pm 1}]$ $\sigma \in TP_2(\mathbb{F}_p)$

Consequences: A any $\mathbb{F}_p$-algebra.

$$\begin{tikzcd} TP(A)/p \arrow[r, "\sim"] & HP(A|\mathbb{F}_p) \arrow[d, leftrightsquigarrow] \\ & \text{de Rham cohom.} \end{tikzcd}$$

ie) $TP(A)$ is a mixed char lift of $HP(A|\mathbb{F}_p)$

Prf: $S^1 \circlearrowright$

$$\begin{tikzcd} THH(A)[2] \arrow[r, "u"] & THH(A) \arrow[r] \arrow[d, rightsquigarrow, "tS^1"] & HH(A|\mathbb{F}_p) \\ TP(A)[2] \arrow[r, "u"] & TP(A) \arrow[r] & HP(A|\mathbb{F}_p) \\ TP(A) \arrow[u, "\cong"] \arrow[ur, "p"'] & & \end{tikzcd}$$

$\square$

Eg) If A smooth $|\mathbb{F}_p$

$\Rightarrow TP(A)$ is lift to $\mathbb{Z}_p$ of something related to de Rham cohom: $\Omega^i_{A|\mathbb{F}_p}$

crystalline cohom. $R\Gamma_{crys}(A|\mathbb{Z}_p) := \Omega^i_{\tilde{A}|\mathbb{Z}_p}$

where $\tilde{A}$ is any smooth alg $/ \mathbb{Z}_p$ lifting A.