Main thm about smooth algs:
(Loday-Quillen, Feigin-Tsygan, Connes)

If A is a smooth k-alg, and $k \supseteq \mathbb{Q}$, then the norm seq. looks like $\prod_{i \in \mathbb{Z}} ?[2i]$ where $?$ is

$$0 \to \Omega^{\geq i}_{A|k} \to \Omega^{\bullet}_{A|k} \to \Omega^{<i}_{A|k} \to 0$$

ie) HP $\longleftrightarrow$ de Rham cohom

$HC^- \longleftrightarrow$ Hodge filtration

<u>Proof</u>: Explicit map of chain cmplxs

$$HH(A|k) \xrightarrow{\varepsilon} [A \xleftarrow{0} \Omega^1_{A|k} \xleftarrow{0} \Omega^2_{A|k} \xleftarrow{0} \cdots]$$

$$a_0 \otimes \cdots \otimes a_n \longmapsto \frac{1}{n!} a_0 \, da_1 \wedge \cdots \wedge da_n$$

On $H_n(-)$, this splits $\varepsilon_n : \Omega^n \to HH_n$. $\square$

§ 2. HC etc. in characteristic p.

<u>Thm</u>: R smooth $/\mathbb{F}_p$. Then the classical thm is still true but the filtration is not naturally split.

eg) $HP(R/\mathbb{F}_p)$ has a filtration whose graded pieces are

$$\Omega^{\bullet}_{R/\mathbb{F}_p}[2i] \quad i \in \mathbb{Z}.$$

<u>"Classical"</u> proof: yoga of dble cmplxs.

<u>Today</u>: Analysis of $HP(R/\mathbb{F}_p)$ via perfectish rings
— will generalise to topological case.

<u>Idea</u>: Don't study smooth algs, but instead

quasiregular semiperfect (qrsp) $\mathbb{F}_p$-algs

— big (non-Noeth.)

— but homol. simple

<u>Defn</u>: An $\mathbb{F}_p$-alg A is <u>qrsp</u> if there exist

- a <u>perfect</u> $\mathbb{F}_p$-alg B ($B \xrightarrow{\cong} B,\ b \mapsto b^p$)
- a <u>regular</u> ideal $I \subseteq B$ (I/I^2 is finite proj B/I-mod)

s.t. $B/I = A$.

· Eg of regular ideal: gen. by a regular sequence.

<u>Examples</u> (1) $\mathbb{F}_p[t^{1/p^\infty}]/(t)$

(2) R smooth $\mathbb{F}_p$-alg

$\rightsquigarrow$ its perfection $R_{perf} := \varinjlim_{x \mapsto x^p} R$

$\rightsquigarrow R_{perf} \otimes_R \cdots \otimes_R R_{perf}$

is qrsp

eg) $\mathbb{F}_p[t]_{perf} \otimes_{\mathbb{F}_p[t]} \mathbb{F}_p[t]_{perf}$

$\cong \mathbb{F}_p[t^{1/p^\infty}] \otimes_{\mathbb{F}_p[t]} \mathbb{F}_p[t^{1/p^\infty}]$

$= \mathbb{F}_p[t_1^{1/p^\infty}, t_2^{1/p^\infty}]/(t_1 - t_2)$

Technique: from $\underbrace{qrsp}$ to smooth.

All of our homol. theories

$$\mathcal{F} = HH(-/\mathbb{F}_p), HC(-/\mathbb{F}_p), \dots$$

$$: \mathbb{F}_p\text{-alg} \longrightarrow D(\mathbb{F}_p)$$

Satisfy flat descent

If $S \to S'$ is a faithfully flat map of $\mathbb{F}_p$-algs then

$$\mathcal{F}(S) \xrightarrow{\sim} \mathrm{Tot}\left(\mathcal{F}(S') \rightrightarrows \mathcal{F}(S' \otimes_S S') \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \cdots \right)$$

Eg) R smooth $/\mathbb{F}_p$, then

$$R \to R_{perf}$$

is faithfully flat.

So:

$$F(R) \xrightarrow{\sim} \mathrm{Tot}\Big(F(R_{perf}) \rightrightarrows F(R_{perf} \otimes_R R_{perf}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots\Big)$$

smooth

qrsp

$\Rightarrow$ Must understand HC^-, HP, HC of any qrsp $\mathbb{F}_p$-alg.

Let A be qrsp.

<u>Step 1</u>: $HH_{odd}(A/\mathbb{F}_p) = 0$,

and $HH_0 = A$

$HH_2 = I/I^2$ (where $A = B/I$)

$HH_{2n} = \Gamma^n_A(I/I^2)$

n^{th} divided power of I/I^2

$\cong \mathrm{Sym}^n_A(I/I^2)$

but multi. is twisted

by $\dfrac{(n+m)!}{n!\,m!}$

$\Rightarrow HH_{2*}(A|\mathbb{F}_p) \cong \Gamma^*_A(I/I^2)$

(Key words: cotangent complex)

<u>Step 2</u>: $HP_0(A|\mathbb{F}_p)$ is a filtered ring with associated graded

$$\Gamma^*_A(I/I^2)$$

What is it?

Answer: The ~~p~~divided power envelope (completed) of

$B \twoheadrightarrow A$. ie) add $f^n/n!$ $\forall f \in I$

Conclusion:

$HP(R/\mathbb{F}_p)$ is built from copies ~~from~~ of

smooth

$$\mathrm{Tot}\Big(\underbrace{HP_0(R_{perf}/\mathbb{F}_p)}\rightrightarrows\underbrace{HP_0(R_{perf}\otimes_R R_{perf}/\mathbb{F}_p)}\cdots\Big)$$

controlled by divided power envelopes.

Also show up in theory of derived de Rham cohomology (Bhatt)

So this Tot is $\simeq \Omega^{\bullet}_{R/\mathbb{F}_p}$.