Topological Hochschild Homology in Arithmetic Geom.

Goals (i) Classical Hochschild | cyclic homology

(ii) "Topological" versions

(iii) Relⁿ to Alg | Arith geom.

Today: <u>§1 Classical theory</u>

Fix comm. base ring k.

For any k-alg A, have Hochschild complex

$$HH(A|k) := \quad A \leftarrow A \otimes_k A \leftarrow A \otimes_k A \otimes_k A \cdots$$

$$a_0 a_1 - a_1 a_0 \leftarrow\!\shortmid a_0 \otimes a_1$$

$$a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 + a_2 a_0 \otimes a_1 \leftarrow\!\shortmid a_0 \otimes a_1 \otimes a_2$$

Hochschild homology $HH_n(A|k)$
$n \geq 0$ are hom. of $HH(A|k)$.

Examples / basic properties

1. $HH_0(A|k) = A/\langle ab - ba \rangle$
$= A/[A,A]$
$= A$ (if A is comm.)

2. If A is comm. then

$HH_1(A|k) = A \otimes_k A / \langle ab \otimes c - a \otimes bc + ac \otimes b : a,b,c \in A \rangle$

Leibnitz

$= \Omega^1_{A|k} \quad adb \longleftrightarrow a \otimes b$

3. $HH_*(A|k) := \bigoplus_{n \geq 0} HH_n(A|k)$
is a comm. graded k-alg (A-alg if

A is comm.

4. 1.–3. $\Longrightarrow \exists$ maps $\varepsilon_n : \Omega^n_{A|k} \longrightarrow HH_n$ by universal property of $\Omega^*_{A|k} := \Lambda^*_A \Omega^1_{A|k}$

<u>Thm</u> (Hochschild–Kostant–Rosenberg 60s)

If A is smooth $|k$ then the maps
$$\varepsilon_n : \Omega^n_{A|k} \longrightarrow HH_n(A|k)$$
are isoms.

Philosophy (Connes, Feigin–Tsygan, Loday–Quillen): Think of HH_* as gen. of diff forms (even if A is non comm.)

To prove HKR, adopt homol. perspective on HH:

Lemma: For any flat k-alg A,

$$HH(A|k) \simeq A \overset{\mathbb{L}}{\underset{A \otimes_k A^{op}}{\otimes}} A$$

Proof: Explicit isom. of complexes

$$HH(A|k) \cong A \overset{\mathbb{L}}{\underset{A \otimes_k A^{op}}{\otimes}} \underbrace{\left[A \otimes_k A \leftarrow A \otimes_k A \otimes_k A \cdots \right]}_{}$$

Bar complex, it is a resol. of A by flat $A \otimes_k A^{op}$-mods. $\square$

Corol:

$$HH_*(A|k) \cong \mathrm{Tor}_*^{A \otimes A^{op}}(A, A)$$

<u>Pf of HKR Thm</u> : A smooth k-alg.

Must show that $HH_*(A|k)$ is the exterior alg on its deg 1 elements.

— this is well-known for the graded alg

$$\mathrm{Tor}_*^B(C,C)$$

when $B \twoheadrightarrow C$ has kernel is locally gen by a reg. seq:

eg) $A \otimes_k A \to A$ (since $k \to A$ is smooth)

Next: cyclic homology.

$$HH(A|k) = A \leftarrow A \otimes_k A \leftarrow A \otimes_k A \otimes_k A \cdots$$

with $\mathbb{Z}/1 \circlearrowright A$, $\mathbb{Z}/2 \circlearrowright A \otimes_k A$, $\mathbb{Z}/3 \circlearrowright A \otimes_k A \otimes_k A$

where

$$\mathbb{Z}/_{n+1} \curvearrowright A^{\otimes n+1}$$

$\Downarrow$

$$\text{gen. } t_n : a_0 \otimes \cdots \otimes a_n \longmapsto a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$$

Set: "Norm" $N := \sum_{i=0}^{n} ((-1)^n t_n)^i : A^{\otimes n+1} \longrightarrow A^{\otimes n+1}$

"Extra degeneracy"

$$s : A^{\otimes n} \longrightarrow A^{\otimes n+1}$$

$$a_0 \otimes \cdots \otimes a_n \longmapsto 1 \otimes a_0 \otimes \cdots \otimes a_n$$

"Connes operator" $B : A^{\otimes n} \xrightarrow{N} A^{\otimes n} \xrightarrow{s} A^{\otimes n+1} \xrightarrow{id - (-1)^n t_n} A^{\otimes n+1}$

<u>Check</u>: $B^2 = 0$, $Bb = -bB$

(where b is the bdry map in HH)

"mixed complex"

or "algebraic S^1-complex"

ie) $B : HH(A|k) \to HH(A|k)[-1]$

<u>Idea</u>: This refines the de Rham diff

$$\begin{array}{ccc} HH_n(A|k) & \xrightarrow{B} & HH_{n+1}(A|k) \\ \uparrow{\scriptstyle \varepsilon_n} & \circlearrowright & \uparrow{\scriptstyle \varepsilon_{n+1}} \\ \Omega^n_{A|k} & \xrightarrow{d} & \Omega^{n+1}_{A|k} \end{array}$$

Defⁿ:

$$
\begin{tikzcd}
 & \vdots \arrow[d, dash, dotted] & \vdots \arrow[d, dash, dotted] & \vdots \arrow[d, dash, dotted] & \vdots \arrow[d, dash, dotted] & \cdots \\
\cdots & A^{\otimes 4} \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "B"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] & \\
\cdots & A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] & & \\
\cdots & A^{\otimes 2} \arrow[d, "b"] & A \arrow[l, "B"'] & & & \\
\cdots & A & & & &
\end{tikzcd}
$$

bideg $(0,0)$

$\underbrace{\qquad\qquad\qquad}_{x \leq 0}$ $\underbrace{\qquad\qquad\qquad}_{x \geq 0}$

$HP(A|k) :=$ product totalisation of this
periodic cyclic hom

able complex

$HC(A|k) :=$ tot. of $x \geq 0$
cyclic hom

$HC^{-}(A|k) :=$ tot of $x \leq 0$
negative cyclic hom.

$$0 \to HH \to HC \xrightarrow{S} HC[2] \to 0$$

$$0 \to HC^-[2] \xrightarrow{S} HC^- \to HH \to 0$$

<u>Norm</u> <u>sequence</u>: $0 \to HC^- \to HP \to HC[2] \to 0$

$$HP \cong \varprojlim \left(\cdots HC[-4] \xrightarrow{S} HC[-2] \xrightarrow{S} HC \right)$$

$$\Rightarrow S : HP \xrightarrow{\cong} HP[2]$$

$$\left(HP_n(Alk) \cong HP_{n+2}(Alk) \right)$$

"Coarse info. about HH gives coarse info about HP, HC^-, HC"

<u>Example</u>: Assume $HH_{odd}(A|k)=0$.
(eg) A perfectoidish)

Then $HP_0(A|k)$ is a complete filtered ring which encodes a lot of the above data.

More precisely, $HP_0(A|k)$ is a ring with filtr by ideals

$$\mathrm{Fil}^n HP_0(A|k) := S^n\left(HC^-_{2n}(A|k)\right)$$

$n \geq 0$

s.t. $$HP_0(A|k)/\mathrm{Fil}^n \cong HC_{2n-2}(A|k)$$

and

$$\mathrm{gr}^n \cong HH_{2n}(A|k)$$

$A = k[G]$.