Let $k = \bar{k}$ be an alg. closed field

$$G \xrightarrow[\substack{\text{smooth,}\\ \text{affine,}\\ \text{conn. fibers}}]{} X \xrightarrow[\text{curve}]{} \mathrm{Spec}(k)$$

<u>Theorem</u> (Gaitsgory–L)

If generic fiber of G is semisimple + simply connected

$$\pi_*(\mathrm{Bun}_G(X))_{\mathbb{Q}_\ell} \xrightarrow{\sim} H^*(X; \mathcal{F}_{BG_X/X})$$

$$H^*(\mathcal{F}_{BG_x/x})_x = \underline{\pi}_*(BG)_x.$$

$$\mathrm{Ran}_G(X) = \left\{ (S, \mathcal{P}, \gamma) : \begin{array}{l} S \subseteq X \\ \text{finite nonempty} \\ \mathcal{P} \text{ a } G\text{-bundle on } X, \ \gamma \text{ trivialization} \\ \text{of } \mathcal{P}|_{X-S} \end{array} \right\} \Big/ \mathrm{iso}$$

$$\mathrm{Ran}_G(X) : \{ \text{comm. } k\text{-algebras} \} \longrightarrow \mathrm{Sets}$$

$$\mathrm{Ran}_G(X)(R) = \left\{ (S, \mathcal{P}, \gamma) : \begin{array}{l} S \subseteq X(R) \text{ finite nonempty} \\ \mathcal{P} \text{ a } G\text{-bundle on } X_R \\ \gamma \text{ trivialization } \\ \text{of } \mathcal{P}|_{X_R - S} \end{array} \right\} \Big/ \mathrm{iso}$$

$$\mathrm{Ran}_G(X) \longrightarrow \mathrm{Bun}_G(X)$$
$$(S, \mathcal{P}, \gamma) \longmapsto \mathcal{P}.$$

set-valued functor

not a sheaf for Zariski-topology.

Can talk about $H^*(\mathrm{Ran}_G(X); \mathbb{Q}_\ell)$ defined as a coh. of a chain complex

$$C^*(\mathrm{Ran}_G(X); \mathbb{Q}_\ell) := \varinjlim C^*(\mathrm{Spec}(R); \mathbb{Q}_\ell)$$

$\mathrm{Spec}(R) \to \mathrm{Ran}_G(X)$.

Theorem (Nonabelian Poincare Duality)

$$\mathrm{Ran}_G(X) \xrightarrow{\theta} \mathrm{Bun}_G(X)$$

induces an isomorphism on ℓ-adic cohomology.

Even better: θ is an "acyclic quasi-fibration"

If

$$\begin{tikzcd}
F \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Spec}(R) \arrow[d, "\mathcal{P} \text{ on } X \times \mathrm{Spec}(R)"] \\
\mathrm{Ran}_G(X) \arrow[r] & \mathrm{Bun}_G(X)
\end{tikzcd}$$

$$F \longrightarrow \mathrm{Spec}(R)$$

induces iso on $\mathbb{Q}_\ell$ cohomology.

For simplicity, assume P is trivial.

<u>Want</u>: $\mathrm{Spec}(R) \times \mathrm{fib}(\theta) \longrightarrow \mathrm{Spec}(R)$ iso on ℓ-adic cohomology.

<u>Want to show</u>:

$\mathrm{fib}(\theta)$ has trivial ℓ-adic cohomology.

$\mathrm{fib}(\theta) = \cancel{\mathrm{Rat}}\ \mathrm{Rat}(X, G).$

$$\mathrm{Rat}(X, G) = \left\{ (S, \gamma) : \begin{array}{l} S \subseteq X \text{ non empty finite} \\ \gamma : X - S \longrightarrow G \end{array} \right\}$$

rational maps from X to G.

Ex: $G = GL_n \times X$.

$\mathrm{Rat}(X, G)$ "=" $\left\{ \begin{array}{l} n\times n \text{ matrices of} \\ \text{rational functions on } X \\ \text{w/ nonvanishing determinant} \end{array} \right\}$

Fix a effective divisor $D \subseteq X$ $\quad D \gg 0$

$\mathbb{A}^M$ = $n\times n$ matrices of elements of $H^0(X, \mathcal{O}(D))$. ← affine space

$$\begin{array}{ccc} \mathbb{A}^M & & \\ \downarrow & & \downarrow \det \leftarrow \text{nonlinear} \\ \mathbb{A}^{1+n\deg(D)-g} & = & H^0(X, \mathcal{O}(nD)) \end{array}$$

infinitely connected

highly connected

$$
\begin{tikzcd}
 & \mathrm{Rat}(X, GL_n) & \\
U \arrow[ur] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \subseteq & \mathbb{A}^M \arrow[d] \\
\mathbb{A}^{1+n\deg(D)-g} - \{0\} \arrow[rr] & & \mathbb{A}^{1+n\deg(D)-g}
\end{tikzcd}
$$

<u>Expectation</u>:

complement of U has codimension $1 + n\deg(D) - g$

$\gtrsim n\deg(D)$.

$$\mathrm{Ran}(X) = \left\{ \begin{array}{c} \text{nonempty finite} \\ \text{subsets } S \subseteq X \end{array} \right\}.$$

$$\mathrm{Ran}(X) = \mathrm{Ran}_{\mathrm{red}}(X).$$

$$\begin{tikzcd}
 & (S,\mathcal{P},\gamma) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
S & \mathrm{Ran}_G(X) \arrow[dl, "\psi"'] \arrow[dr, "\theta"] & \mathcal{P} \\
\mathrm{Ran}(X) & & \mathrm{Bun}_G(X)
\end{tikzcd}$$

Fix a point of $\mathrm{Ran}(X)$

$\{x\} \in \mathrm{Ran}(X) \ni S$

$$\psi^{-1}\{x\} \in \mathrm{Ran}_G(X)$$

$$\begin{array}{ccc} \psi^{-1}\{x\} & & \psi^{-1}(S) \\ \| & & \| \\ \left\{ \begin{array}{l} (\mathcal{P}, \gamma): \\ \mathcal{P} \text{ is a } G\text{-bundle on } X,\ \gamma \\ \text{trivialization on } X - \{x\} \end{array} \right\} & & \left\{ \begin{array}{l} (\mathcal{P}, \gamma) \ldots \\ \text{-- -- --} \\ \ldots \text{ on } X - S \end{array} \right\} \\ \overset{!!}{=} & & \| \\ \mathrm{Gr}_{G,x}. & & \prod_{x \in S} \mathrm{Gr}_{G,x}. \end{array}$$

local object.

Factorization property.

$$H^*(\mathrm{Bun}_G(X)) \simeq H^*(\mathrm{Ran}_G(X))$$

is

$$H^*(\mathrm{Ran}(X); \underbrace{R\psi_* \mathbb{Q}_\ell}_{\mathcal{A}}).$$

Assume that G is everywhere semisimple.

Then $\mathrm{Gr}_{G,x}$ is an Ind-proper variety (for all $x \in X$)

$\psi : \mathrm{Ran}_G(X) \to \mathrm{Ran}(X)$
is "proper".

Proper base change:

$$H^*(\mathcal{A}_{\{x\}}) = H^*(Gr_{G,x})$$

$$H^*(\mathcal{A}_S) = \bigotimes_{x \in S} H^*(Gr_{G,x}).$$

$\mathcal{A}$ is a factorizable sheaf.

$$H^*(Ran(X); \mathcal{A})$$

Factorization (co)homology of X w/ coefficients in $\mathcal{A}$.

$$\cong$$

$$H^*(Bun_G(X); \mathbb{Q}_\ell)$$

Factorization homology is an invariant that makes sense in topology; associated to factorization algebras on manifolds.

Recovers $HH_*(A)$ when the manifold is S^1.

We are doing analogue when S^1 is replaced by an algebraic curve X.

Heuristic Formulation

$$\text{``}\bigotimes_{x\in X}^{\text{cont}} H_*(Gr_{G,x}) \simeq H_*(Bun_G(X))\text{''}$$

$$\boxed{\left(\bigotimes_{x\in S^1}^{\text{cont}} A \simeq HH_*(A/k)\right)}$$

$$\Big\Downarrow \text{ Today}$$

Today

$$\bigotimes_{x\in X}^{\text{cont}} H_*(Gr_{G,x}) \xrightarrow{\ \sim\ } H_*(Bun_G(X))$$

coefficients in $\mathbb{Z}_\ell$

$$\begin{array}{ccc} \Big\updownarrow \text{ Koszul Dual} & \Big\Downarrow & \Big\updownarrow \text{ Vector space duality.} \end{array}$$

Lecture 2

$$\bigotimes_{x\in X}^{\text{cont}} H^*(BG_x) \xrightarrow{\ \sim\ } H^*(Bun_G(X))$$

coefficients in $\mathbb{Z}_\ell$

Lecture 4

$$\Big\Downarrow$$

Formal from definition of $\mathcal{F}_{BG_x/X}$.

$$H^{-*}(X;\mathcal{F}_{BG_x/X}) \xleftarrow{\ \sim\ } \pi_*(Bun_G(X))_{\mathbb{Q}_\ell}$$

$\Downarrow$ Lecture 4

Weil's Conjecture

$$\mathrm{Ran}(X) = \varinjlim_{\substack{S \in \text{finite sets,} \\ \text{surjections}}} X^S$$