$\overline{Y} := Y \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q)$ $\quad Y \longrightarrow \mathrm{Spec}(\mathbb{F}_q)$

smooth of dimension D

$$|Y(\mathbb{F}_q)| = \sum_{\substack{y \in Y(\mathbb{F}_q) \\ \text{(iso classes)}}} \frac{1}{|\mathrm{Aut}(y)|}$$

<u>Def</u> Y <u>satisfies the G-L trace formula</u> if

$$\frac{|Y(\mathbb{F}_q)|}{q^D} = \mathrm{Tr}(\varphi^{-1} \mid H^*(\overline{Y}))$$

$$:= \sum (-1)^i \, \mathrm{Tr}(\varphi^{-1} \mid H^i(\overline{Y}))$$

<u>Ex</u> True if Y is a variety.

Ex: G linear alg. group over $\mathbb{F}_q$

$Y = BG$

$\mathrm{Spec}(R) \to Y = BG$

$\Updownarrow$

Principal G-bundle on $\mathrm{Spec}(R)$

Ex: $G = \mathbb{G}_m$

$Y = B\mathbb{G}_m$.

$Y(\mathbb{F}_q)$ = category of 1-dim'l vector spaces over $\mathbb{F}_q$.

$$|Y(\mathbb{F}_q)| = \frac{1}{q-1}$$

$D = \dim \mathbb{B}\mathbb{G}_m$

$\mathbb{B}\mathbb{G}_m = * // \mathbb{G}_m$

$\dim \mathbb{B}\mathbb{G}_m = -1.$

$$\frac{|Y(\mathbb{F}_q)|}{q^{\dim Y}} = \frac{q}{q-1}$$

<u>RHS</u>: $\mathrm{Tr}(\varphi^{-1} \mid H^*(\overline{\mathbb{B}\mathbb{G}_m}))$

$\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$

$\mathbb{B}\mathbb{G}_m \qquad B\mathbb{C}^\times \leftarrow$ quotient of a contractible space by free action of $\mathbb{C}^\times$.

$V - \{0\} \circlearrowleft \mathbb{C}^*$

$\mathbb{C}$-vector space

V has dimension $n < \infty$

$$V - \{0\} \simeq S^{2n-1}$$

If $\dim V = \infty$

$V - \{0\}$ is contractible

$$B\mathbb{C}^* = (V - \{0\}) / \mathbb{C}^*$$

$$\parallel$$

$$\mathbb{C}P^\infty$$

$H^*(\mathbb{C}P^\infty; A) \simeq A[t]$ $\deg(t) = 2$.

$$H^*(\overline{BG_m}) = \mathbb{Q}_\ell[t]_{\deg(t)=2}.$$

$$\varphi(t) = qt.$$

$$\varphi(t^n) = q^n t^n.$$

$$\mathrm{Tr}(\varphi^{-1} | H^*(BG_m)) = \sum_{n \geq 0} q^{-n} = \frac{q}{q-1}$$

<u>Conclusion</u>: G-L is okay for BG_m.

ℓ-adic homotopy

$\overline{Y}$ ← algebro-geom object over alg. closed field $k \left(= \overline{\mathbb{F}}_q\right)$

$y \in \overline{Y}(k)$

$\pi_1^{et}(\overline{Y}, y)$ ← profinite group.

(Assume $\overline{Y}$ connected)

$\{$ Finite étale covers of $\overline{Y} \}$

$\Updownarrow$

$\{$ finite sets w/ cont. action $\pi_1^{et}(\overline{Y}, y)$ $\}$.

$\pi_1^{et}(\bar{Y}, y)_\ell$ ← maximal pro-ℓ quotient of $\pi_1^{et}(\bar{Y}, y)$

$\ell \neq 0$ in k.

Artin–Mazur (ℓ-adic version)

To $\bar{Y}$, they associate a topological space Z ← ℓ-adic homotopy type of $\bar{Y}$, with

1) Z is simply connected and $\pi_n Z$ is a finitely gen. $\mathbb{Z}_\ell$-module $\forall n$.

2) $H^*_{sing}(Z; \mathbb{Z}/\ell)$ is $H^*_{et}(\bar{Y}; \mathbb{Z}/\ell)$

Assume:

$\pi_1(\bar{Y}, y)_\ell = 0$

⇕

$H^1_{et}(\bar{Y}; \mathbb{Z}/\ell) = 0$

Also:

$H^n_{et}(\bar{Y}; \mathbb{Z}/\ell)$ finite

For each $n > 0$

$$\pi_n(\bar{Y}) := \pi_n(Z)$$

↖ finitely generated $\mathbb{Z}_\ell$-module

$$\pi_n(\bar{Y})_{\mathbb{Q}_\ell} := \pi_n(\bar{Y})[\tfrac{1}{\ell}].$$

↖ finite dim'l vector space over $\mathbb{Q}_\ell$.

Have a canonical pairing

$$b : \pi_n(\bar{Y})_{\mathbb{Q}_\ell} \times H^n(\bar{Y}) \longrightarrow \mathbb{Q}_\ell$$

$f: S^n \to Z$ $\qquad \eta \in H^n(\bar{Y})$

$$b(f, \eta) = f^*\eta \in H^n(S^n; \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell.$$

$$I = H^*_{red}(\bar{Y}) = \bigoplus_{n>0} H^n(\bar{Y})$$

$b: \pi_*(\overline{Y})_{\mathbb{Q}_\ell} \times I \longrightarrow \mathbb{Q}_\ell$

descends to a pairing

$\bar{b}: \pi_*(\overline{Y})_{\mathbb{Q}_\ell} \times I/I^2 \longrightarrow \mathbb{Q}_\ell.$

$\eta = \eta' \eta''$

$f^*(\eta) = f^*(\eta') f^*(\eta'') = 0.$

<u>Assertion</u>: If $H^*(\overline{Y})$ is polynomial ring (on even generators), then $\bar{b}$ is a perfect pairing.

$\cdots \subseteq I^3 \subseteq I^2 \subseteq I \subseteq H^*(\overline{Y})$

<u>Ex</u>: $\overline{Y} = \overline{B\mathbb{G}_m}$, this applies.

Suppose $H^*(\bar{Y})$ is a polynomial ring

$$\bar{Y} = Y \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\bar{\mathbb{F}}_q)$$

$$\mathrm{Tr}(\varphi^{-1} \mid H^*(\bar{Y})) := \sum (-1)^i \mathrm{Tr}(\varphi^{-1} \mid H^i(\bar{Y}))$$

$$(\pi_* \bar{Y}|_{\mathbb{Q}_\ell} \simeq (I/I^2)^\vee$$

← finite dim'l over $\mathbb{Q}_\ell$.

φ has complex eigenvalues

$\lambda_1, \lambda_2, \dots, \lambda_n$. on $\pi_*(\bar{Y})_{\mathbb{Q}_\ell}$.

φ^{-1} has eigenvalues

$\lambda_1 \dots \lambda_n$ $\quad I/I^2$.

$$\mathrm{Tr}(\varphi^{-1} \mid H^*(\bar{Y})) = \mathrm{Tr}(\varphi^{-1} \mid \mathrm{gr}\, H^*(\bar{Y}))$$

$$= \mathrm{Tr}(\varphi^{-1} \mid \mathrm{Sym}^*(I/I^2))$$

$$\mathbb{Q}_\ell \hookrightarrow \underline{\mathbb{C}}.$$

$$\mathrm{Tr}\left(\varphi^{-1} \mid \mathrm{Sym}^*(I/I^2)\right)$$

$$\| $$

$$\sum_{e_1, e_2, \ldots, e_n \geq 0} \lambda_1^{e_1} \lambda_2^{e_2} \cdots \lambda_n^{e_n}.$$

$$\|$$

$$\prod_{i=1}^{n} \frac{1}{1-\lambda_i}.$$

$$\mathrm{Tr}\left(\varphi^{-1} \mid H^*(\overline{Y})\right) = \left(\det\left(1-\varphi \mid (\pi_* \overline{Y})_{\mathbb{Q}_\ell}\right)\right)^{-1}$$

Ex $\overline{Y} = B\mathbb{G}_m$. $(\pi_* \overline{Y})_{\mathbb{Q}_\ell}$ ← 1-dim'l vector space.

$\circlearrowright$ $\varphi = \frac{1}{q}$.

$$= \det(1-\varphi)$$
$$= (1-\tfrac{1}{q})$$

Ex: Let G be **any** connected linear algebraic group over $\mathbb{F}_q$.

G-L trace formula for BG.

$$\frac{|BG(\mathbb{F}_q)|}{q^{\dim(BG)}} \overset{?}{=} \mathrm{Tr}\left(\varphi^{-1} \mid H^*(\overline{BG})\right)$$

$$\|\qquad\qquad\qquad\qquad\|$$

$$\frac{q^{\dim(G)}}{|G(\mathbb{F}_q)|} \overset{?}{=} \left(\det\left(1-\varphi \mid \pi_*(\overline{Y})_{\mathbb{Q}_\ell}\right)\right)^{-1}$$

Steinberg's Formula

$$|G(\mathbb{F}_q)| = q^{\dim(G)} \det\left(1-\varphi \mid \pi_*(\overline{BG})_{\mathbb{Q}_\ell}\right)$$

Ex: $G = GL_n$

$$H^*(B\overline{G}) = \mathbb{Q}_\ell[c_1, c_2, \ldots, c_n]$$

$$\pi_*(B\overline{G})_{\mathbb{Q}_\ell} = \mathbb{Q}_\ell\{e_1, e_2, \ldots, e_n\}$$

$$u(c_i) = q^i c_i$$

$$u(e_i) = q^{-i} e_i$$

Steinberg:

$$|GL_n(\mathbb{F}_q)| = q^{n^2} \cdot \left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{q^2}\right) \cdots \left(1 - \frac{1}{q^n}\right)$$

In general, (not assuming $H^*(\bar{Y})$ is polynomial)

there is a spectral sequence

$$\mathrm{Sym}^*(\pi_* \bar{Y})^\vee_{\mathbb{Q}} \Rightarrow H^*(\bar{Y})$$

Gives same conclusion

$$\mathrm{Tr}(\varphi^{-1} \mid H^*(\bar{Y})) = \left(\underline{\det(1-\varphi \mid \pi_*(\bar{Y}))}\right)^{-1}$$

$$:= \prod_i \det(1-\varphi \mid \pi_i(\bar{Y}))^{(-1)^{i+1}}$$

assuming everything converges.

(For example, if $\pi_*(\bar{Y})_{\mathbb{Q}_\ell}$ is finite dim'l.

This will apply when

$$Y = \mathrm{Bun}_G(X).$$