$X \to \mathrm{Spec}(\mathbb{F}_q)$

smooth proj. curve over $\mathbb{F}_q$

finite field

K_X ← fraction field of X.

Analogy

Number Fields	Function Fields
$\mathbb{Q}$	K_X
prime numbers p (+ point at ∞)	closed points $x \in X$
$\mathbb{Z}/p\mathbb{Z}$	$k(x) \leftarrow$ residue field at x.
$\mathbb{Z}_p$	$\mathcal{O}_x \leftarrow$ complete local ring of X at x $\mathcal{O}_x \cong k(x)[[t]]$
$\mathbb{Q}_p$, (or $\mathbb{R}$)	$K_x \approx k(x)((t))$
$\mathbb{A}$	$\mathbb{A}_X = \prod^{\text{res}}_{x \in X} K_x$
quadratic form q_0 over $\mathbb{Q}$ (SO_{q_0})	semisimple group G_0 over K_X.
$SO_{q_0}(\mathbb{Q}) \subseteq SO_{q_0}(\mathbb{A})$	$G_0(K_X) \subseteq G_0(\mathbb{A}_X)$
μ_{Tam}	μ_{Tam}
$\mu_{Tam}(Spin_{q_0}(\mathbb{Q}) \backslash Spin_{q_0}(\mathbb{A}))$ $\shortparallel$ 1	$\mu_{Tam}(G_0(K_X) \backslash G_0(\mathbb{A}_X))$ $\shortparallel$ 1

q quadratic form over $\mathbb{Z}$

$SO_q(\mathbb{Z}/p\mathbb{Z})$

$$\mathrm{Mass}(q) = \sum_{q' \text{ genus of } q} \frac{1}{|O_{q'}(\mathbb{Z})|}$$

Mass Formula

group scheme

$$G \longrightarrow X$$

(Ex: $G = X \times GL_n$
$G = X \times SL_n$)

$G(K(x))$

$$\sum_{\substack{\text{Principal} \\ G\text{-bundles} \\ P \text{ on } X}} \frac{1}{|\mathrm{Aut}(P)|}$$

Mass Formula

$$\sum_P \frac{1}{|\mathrm{Aut}(P)|} = q^D \cdot \prod_{x \in X}$$

$d := \dim(G_0 / K_x)$

$D := \dim \mathrm{Bun}_G(X$

$\mathrm{Bun}_G(X) \longleftarrow$ moduli *stack* of G-bundles

$$\text{Maps}\ \mathrm{Spec}(R) \longrightarrow \mathrm{Bun}_G(X)$$

$$\wr\wr$$

$$G\text{-bundles on } X \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(R)$$

Goal:

Compute

$$\sum \frac{1}{|\mathrm{Aut}(P)|} =: \left| \mathrm{Bun}_G(X)(\mathbb{F}_q) \right|$$

Digression:

Y alg. variety over $\mathbb{F}_q$.

$$|Y(\mathbb{F}_q)| \qquad \bar{Y} := Y \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\bar{\mathbb{F}}_q)$$

Idea: Think of

$$Y(\mathbb{F}_q) \subseteq \bar{Y}$$

$$\begin{array}{ccc} \bar{Y} & \xrightarrow{\varphi} & \bar{Y} \\ \cap\!| & & \cap\!| \\ \mathbb{P}^n & \xrightarrow{\varphi} & \mathbb{P}^n \\ {[x_0 : \dots : x_n]} & & {[x_0^q : \dots x_n^q]} \end{array}$$

$Y(\mathbb{F}_q) =$ fixed points of φ.

$\ell \neq 0$ in $\mathbb{F}_q$.

<u>Idea</u> (Weil)

$|Y(\mathbb{F}_q)|$ should be

ℓ-adic cohomology of $\overline{Y}$

$$\sum_{i \geq 0} (-1)^i \operatorname{Tr}(\varphi \mid H^i_c(\overline{Y}))$$

Theorem (Grothendieck-Lefschetz Formula)

Assume Y smooth of dimension d,

$H^i_c(\overline{Y}) \approx H^{2d-i}(\overline{Y})^\vee$ (Poincaré Duality)

not φ-equivariant.

$$\sum (-1)^i \operatorname{Tr}(\varphi^{-1} \mid H^i(\overline{Y}))$$

$$\| \qquad \frac{|Y(\mathbb{F}_q)|}{q^d}$$

Idea: Apply this to $Y = \mathrm{Bun}_G(X)$.

Def: $\mathrm{Bun}_G(X)$ satisfies the G-L trace formula if

$$\frac{\sum \frac{1}{|\mathrm{Aut}(P)|}}{q^{\dim(\mathrm{Bun}_G(X))}} = \sum (-1)^i \, \mathrm{Tr}(\varphi^{-1} | H^i(\overline{\mathrm{Bun}_G(X)}$$

$$|| $$

$$\mathrm{tr}(\varphi^{-1} | H^*(\overline{\mathrm{Bun}_G(X)}$$

Weil's conjecture follows from two assertions

1) $\mathrm{Bun}_G(X)$ satisfies GL.

$$\frac{\sum \frac{1}{|\mathrm{Aut}(\mathcal{P})|}}{q^D}$$

Theorem of Behrend in case G is a constant group (generalized)

$$\|$$

2)
$$\boxed{\begin{gathered}\mathrm{tr}\left(\varphi^{-1} \,\middle|\, H^*(\overline{\mathrm{Bun}_G(X)})\right) \\ \| \\ \prod_{x \in X} \left(\frac{|G(\kappa(x))|}{|\kappa(x)|^d} \right)^{-1}\end{gathered}}$$

Digression

Let $x \in X$ be closed point.

$$\mathrm{Bun}_G(\{x\}) = BG_x.$$

$$\mathrm{Bun}_G(\{x\})(\mathbb{F}_q)$$
$$\|$$
$$\left\{ \begin{array}{c} \text{principal } G\text{-bundles} \\ \text{on } \mathrm{Spec}(\kappa(x)) \end{array} \right\}$$

has one object, symmetry group is $G(\kappa(x))$.

$$\frac{|\mathrm{Bun}_G(\{x\})(\mathbb{F}_q)|}{q^{\dim \mathrm{Bun}_G(\{x\})}} = \frac{|\kappa(x)|^d}{|G(\kappa(x))|}$$

$\mathrm{Bun}_G(\{x\})$ ← satisfy GL trace formula.

$$\frac{|k(x)|^d}{|G(k(x))|} = \mathrm{tr}(\mathrm{Ce}^{-1} \mid H^*(\overline{\mathrm{Bun}_G(\{x\})})$$

$$\mathrm{tr}(\mathrm{Ce}^{-1} \mid H^*(\mathrm{Bun}_G(X)))$$

$\overset{?}{=}$ ← Weil's Conjecture.

$$\prod_{x \in X} \mathrm{tr}(\mathrm{Ce}^{-1} \mid H^*(\mathrm{Bun}_G(\{x\})))\;).$$

$$\mathrm{Bun}_G(X) = \text{"}\prod_{x\in X}^{\mathrm{cont}} \mathrm{Bun}_G(\{x\})\text{"}$$

$$H^*\left(\overline{\mathrm{Bun}_G(X)}\right) = \text{"}\bigotimes_{x\in X}^{\mathrm{cont}} H^*\left(\overline{\mathrm{Bun}_G(\{x\})}\right)\text{"}$$

Makes sense using theory of factorization homology

$$\prod_{x \in X} \frac{1}{1 - \frac{1}{|\kappa(x)|^2}}$$

$$\frac{|SL_2(\mathbb{F}_q)|}{q^{\dim}} = \frac{q^3 - q}{q^3}$$

$$= 1 - \frac{1}{q^2}$$

$$\mathrm{Bun}_G(X) \qquad G = \mathbb{G}_m.$$
$$\|$$
$$\coprod_{n \in \mathbb{Z}} \mathrm{Bun}_G^{\deg(n)}(X)$$