x^2+y^2 x^2-y^2 $-x^2-y^2$

x^2+y^2 x^2+3y^2

(Positive Def)

<u>Def</u>: Quadratic Forms q and q' are <u>in same genus</u> if they are $\simeq$ mod N for all $N>0$.

If q a form over $\mathbb{Z}$.

R comm. ring

$$O_q(R) = \{ A \in GL_n(R) \mid q \circ A = q \}$$

$$O_q(\mathbb{R}) \leftarrow \text{compact Lie group.}$$

$$\cup\!|$$

$$O_q(\mathbb{Z})$$

$$\mathrm{Mass}(q) = \sum_{\substack{q' \text{ of} \\ \text{genus } (q)}} \frac{1}{|O_{q'}(\mathbb{Z})|}$$

Def

q is __unimodular__

if nondegenerate (mod p)

for all p.

$$x^2 + y^2 \equiv (x+y)^2 \pmod 2.$$

Mass Formula (Unimodular Case)

$8 \mid n$

$$\mathrm{Mass}(q) = \text{something else.}$$

$$\Vert$$

$$\sum_{q \text{ unimodular}} \frac{1}{|O_q(\mathbb{Z})|} = \frac{\cancel{\zeta(\tfrac{n}{2})}\,\zeta(2)\,\zeta(4)\cdots\zeta(n-2)\,\cancel{\zeta(}}{\mathrm{vol}(S^1)\,\mathrm{vol}(S^2)\cdots\mathrm{vol}(S^{n-1})}$$

__Ex__: $n=8$

__RHS__ $= \dfrac{1}{2^{14} \cdot 3^5 \cdot 5^2 \cdot 7}$

$|W(E_8)|$

__Mass Formula__

$\Updownarrow$

$\exists!$ unimodular form in 8 variables

<u>Ex</u> $n=32$

<u>RHS</u> $\approx 40{,}000{,}000$

$\Rightarrow \exists$ millions of inequivalent unimodular forms in 32 variables

Let q, q' are in same genus

$$q = q' \circ A_N \qquad A_N \in GL_n(\mathbb{Z}/N\mathbb{Z})$$

WLOG, $\{A_N\} = A \in GL_n(\hat{\mathbb{Z}})$

$$\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/N\mathbb{Z}$$
$$= \prod_p \mathbb{Z}_p.$$

$q = q' \circ A \Longrightarrow q, q'$ are equivalent over $\mathbb{Z}_p$ for all p

$\Longrightarrow$ - - - - over $\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}]$

Hosse-Minkowski

$\Longrightarrow$ $q = q' \circ B$

$B \in GL_n(\mathbb{Q})$.

$$q = q' \circ A = q \circ B^{-1} \circ A$$

$\mathbb{Q} \cap \hat{\mathbb{Z}} = \mathbb{Z}$

$$\mathbb{A}^{fin} = \hat{\mathbb{Z}} \otimes \mathbb{Q}$$
$$\subseteq \prod \mathbb{Q}_p$$
$$\mathbb{A} := \mathbb{A}^{fin} \times \mathbb{R}.$$

$$B^{-1} \circ A$$
$$\cap$$
$$O_q(\mathbb{Q}) \backslash O_q(\mathbb{A}) / O_q(\hat{\mathbb{Z}} \times \mathbb{R})$$

Want to count size of this.

$$\mathbb{Q} \subseteq \mathbb{A}$$

locally compact ring

discrete subring.

$$\mathcal{O}_q(\mathbb{Q}) \subseteq \overset{\mu}{\mathcal{O}_q(\mathbb{A})} \supseteq \mathcal{O}_q(\hat{\mathbb{Z}} \times \mathbb{R})$$

discrete subgroup

locally compact group

compact open subgroup?

$$\mu$$

$$\mathcal{O}_q(\mathbb{Q}) \backslash \mathcal{O}_q(\mathbb{A}) \circlearrowleft \mathcal{O}_q(\hat{\mathbb{Z}} \times \mathbb{R})$$

\# of ~~orbits~~ $= \dfrac{\mu\left(\mathcal{O}_q(\mathbb{Q}) \backslash \mathcal{O}_q(\mathbb{A})\right)}{\mu\left(\mathcal{O}_q(\hat{\mathbb{Z}} \times \mathbb{R})\right)}$

$\mathrm{Mass}(q) =$

$$SO_q(\mathbb{R}) = \left\{ A \in Gl_n(\mathbb{R}) \;\middle|\; \begin{matrix} q \circ A = q \\ \det(A) = 1 \end{matrix} \right\}$$

$$\mu'$$

$$SO_q(\mathbb{A})$$

$$2^k \, \mathrm{mass}(q) = \frac{\mu'\left(SO_q(\mathbb{Q}) \backslash SO_q(\mathbb{A}) \right)}{\mu'\left(SO_q(\hat{\mathbb{Z}} \times \mathbb{R}) \right)}$$

$SO_q(\mathbb{A})$ has a <u>canonical</u> Haar measure called <u>Tamagawa measure</u>.

$$\mu_{Tam} := \prod_p \underline{\mu_{\omega, \mathbb{Q}_p}} \times \underline{\mu_{\omega, \mathbb{R}}}.$$

independent of ω.

$$SO_q(\mathbb{A}) = SO_q(\mathbb{R}) \times \prod_p^{\mathrm{res}} SO_q(\mathbb{Q}_p)$$

$$\begin{tikzcd}
& \mu_{\omega,\mathbb{R}} & V_{\mathbb{R}} \\
\omega \in V_{\mathbb{Q}},\ \omega \neq 0 \arrow[ur, mapsto] \arrow[dr, mapsto] & & V_{\mathbb{Q}} \arrow[u, phantom, sloped, "\subset"] \arrow[d, phantom, sloped, "\subset"] \\
& \mu_{\omega,\mathbb{Q}_p} & V_{\mathbb{Q}_p}
\end{tikzcd}$$

$V_{\mathbb{R}}$ = space of translation-invariant top forms on $SO_q(\mathbb{R})$.

$V_{\mathbb{Q}}$ = space of ... algebraic top forms on SO_q.

$V_{\mathbb{Q}_p}$ = ... $SO_q(\mathbb{Q}_p)$

$SO_q(\mathbb{Q}_p)$ ← p-adic analytic Lie group

$$\mathrm{Mass}(q) = 2^{-k} \frac{\mu_{Tam}(SO_q(\mathbb{Q})\backslash SO_q(\mathbb{A}))}{\mu_{Tam}(SO_q(\hat{\mathbb{Z}} \times \mathbb{R}))}$$

$$SO_q(\hat{\mathbb{Z}} \times \mathbb{R}) = SO_q(\mathbb{R}) \times \prod_p SO_q(\mathbb{Z}_p)$$

$$\mu_{Tam}(SO_q(\hat{\mathbb{Z}} \times \mathbb{R}))$$

$$\|\mathrm{def}$$

$$\mu_{\omega,\mathbb{R}}(SO_q(\mathbb{R})) \times \prod_p \mu_{\omega,\mathbb{Q}_p}(SO_q(\mathbb{Z}_p))$$

Mass Formula (Tamagawa-Weil Version)

$$\mu_{Tam}(SO_q(\mathbb{Q})\backslash SO_q(\mathbb{A})) = 2.$$

$$\mathrm{Spin}_q \longrightarrow SO_q$$

Equivalent:

$$\mu_{Tam}\left(\mathrm{Spin}_q(\mathbb{Q}) \backslash \mathrm{Spin}_q(\mathbb{A})\right) = 1$$

Conjecture (Weil) (Proved by Langlands, Lai, Kottwitz)

Let G be a simply connected semisimple alg. group over $\mathbb{Q}$.

$$\mu_{Tam}\left(G(\mathbb{Q}) \backslash G(\mathbb{A})\right) = 1.$$

$$\underset{\tau_G}{\shortparallel}$$

$\tau_G \leftarrow$ Tamagawa number.