Kronecker-Weber

Abelian Galois extensions of $\mathbb{Q}$ are $\mathbb{Q}[\varepsilon_n]$

n^{th} root of 1.

Lubin-Tate: Get Galois extensions ~~from~~ using formal groups.

Formal gp law over R

$F(X,Y) = X \underset{F}{+} Y = X + Y + \cdots$

$$x +_F 0 = 0 +_F x = x$$

$$x +_F y = y +_F x$$

$$(x +_F y) +_F z = x +_F (y +_F z)$$

Lie variety over $\mathbb{R}$

objects $\mathbb{A}^n$ $\qquad n = 0, 1, 2, \ldots$

maps $\mathbb{A}^n \longrightarrow \mathbb{A}^1$ $\qquad f(x_1 \ldots x_n)$

$$\cap$$

$$\mathbb{R}[[x_1 \ldots x_n]]$$

$$\mathbb{A}^n \xrightarrow{\simeq} \prod^n \mathbb{A}^1$$

$$(x_1 \ldots x_n)$$

<u>Question</u> · how many formal group laws are there?

· how to construct formal group laws?

Thm (Lazard)

$$R \longmapsto \{\text{formal gp laws}/R\}$$

$$\mathrm{Ring}(L, R)$$

$$L = \mathbb{Z}[x_1, x_2 \dots]$$

ISG $\quad F \xrightarrow{g} G$

$g(x)$

$$g(x +_F y) = g(x) +_G g(y)$$

Universal ISG over

$L[s_1, s_2, \ldots]$

Algebraic topology

Cohomology theories E with chern classes for complex line bundles

$$\begin{array}{c} V \\ \downarrow \\ X \end{array} \longmapsto c_i(X) \in E^{2n}(X)$$

$$c_n(V \oplus W) = \sum_{i+j=n} c_i(V)\, c_j(W)$$

not true in general

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_2(L_2)$$

$\dim L_i = 1$

Thm (Quillen) For general E, $\exists$ formal gp law F

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2))$$

$$= c_1(L_1) \underset{F}{+} c_1(L_2)$$

$$H^s(\mathcal{M}_{FG}; \omega^t) \Rightarrow \pi_{2t-s} S^0$$
$$= \lim_{n \to \infty} \pi_{2t-s+n} S^n$$

$\omega = \mathrm{Lie}\, F^*$

Ex G_a $\qquad x+y$

G_m $\qquad x+y-xy$

$$= 1-(1-x)(1-y)$$

Are they iso?

$$g(x) = 1-e^{-x}$$

$$g(x+y) \underset{\text{maybe}}{=} g(x)+g(y) - g(x)g(y)$$

Over $\mathbb{Q}$

Are they iso ~~ ~~ over $\mathbb{F}_p$?

$$g: G_a \to G_m$$

$$g(\underbrace{x+\cdots+x}_{p}) = 1-(1-g(x))^p$$

$$g(0) = g(x)^{\varphi} = g^{\varphi}(x^{p})$$
$$\shortparallel$$
$$0$$

$$\Rightarrow \quad g = 0$$

<u>Height:</u> $R = k$ field of char $p > 0$.

$$f : G_1 \longrightarrow G_2$$

thm $\exists$ unique $g(x)$ $g'(0) \neq 0$

$$q = p^a$$

$$f(x) = g(x^q)$$

$a =$ height of f.

Height of a formal gp G

$:=$ height of mult by p

height $\hat{G}_a = \infty$

height $\hat{G}_m = 1$

<u>Thm</u> (Dieudonné) k perfect alg closed, any two formal gps of the same height are isomorphic.

Lubin-Tate deformation spaces

Γ $\qquad\qquad$ B ← complete local m - max ideal

k

field char $p > 0$

A deformation of Γ to B

$$\begin{tikzcd} & B \arrow[d, "r"'] & G \\ k \arrow[r, "i"'] & B/\mathfrak{m} & \end{tikzcd}$$

(G, i, f) $\qquad$ $G \xrightarrow{f} i^*\Gamma$

~~Deform~~

$$\begin{tikzcd} \mathrm{Deform}_\Gamma(B) \arrow[d, squiggly] & \text{groupoid} \arrow[l] \\ \pi_0 \mathrm{Deform}_\Gamma(B) & \text{iso classes of objects} \end{tikzcd}$$

Thm (L-T) n = height Γ

$$\pi_0 \, \mathrm{Deform}_n(B) = \mathfrak{M}^{n-1}$$

want to understand

$$\mathfrak{M}^{n-1} /\!/ \operatorname{aut} \Gamma$$

G_{univ} universal deformation

$\mathbb{W}[[u_1 \cdots u_{n-1}]]$ $\mathbb{W}$ = Witt vectors of k

universal deformation

$E_0 = W[[u_1 \cdots u_{n-1}]]$

$E_* = W[[u_1 \cdots u_{n-1}]][u, u^{-1}]$

$|u| = -2$

$\text{Aut}\,\Gamma = S_n$ acts on E_*

height n

$u E_0 = E_{-2}$ sections of Lie G

interested in

$H^*(S_n; E_0)$

$H^0(S_n; E_{2k})$

not the symmetric group

<u>Question</u> Can one write down explicitly the action of $\mathrm{Aut}\,\Gamma$ on $W[[u_1 \ldots u_{n-1}]]$.

<u>Questions</u> what is Pic (LT-space)

$$= H^1(\mathrm{Aut}\,\Gamma ; E_0^*)$$

conjectured answer exists
known $n=2$ $p>5$

Observation

2 $n=2$

$$H^*(S_n; W) \xrightarrow{\cong} H^*(S_n; E_0)$$

$p>3$ Shimomura

$p \leq 3$ ~~Beaud~~ Beaudry, Bobkova, Behrens, Henn

True $n>2$?