IMC for $E/\mathbb{Q}_\infty$

$L(E,1) = 0 \iff \mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$ has corank ≥ 1

$L(E,1) \neq 0$ get p-part of the BSD formula

$L'(E,1) \neq 0$?

$\mathrm{crk}\, \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = 1$?

GZ formula

$E \quad N_E \qquad \mathrm{ord}_{s=1} L(E,s) = 1$

K imag. quad. field.

- $\ell \mid N_E \Rightarrow \ell$ split in K
- $L(E^K,1) \neq 0$

$$\mathrm{ord}_{s=1} L(E/K, s) = 1$$
$$L(E/K,s) = L(E,s)\,L(E^K,s)$$

GZ formula $\qquad y_K \in E(K)$ Heegner pt.

$$\frac{L'(E/K,1)}{\Omega_{E/K}\sqrt{|D_K|}} = (*) \langle y_K, y_K \rangle_{NT}$$

Also

rk $E(K) = 1$ $\quad \mathbb{Z} y_K \subseteq E(K)$

finite index

$$\frac{L'(E/K,1)}{\sqrt{|D_K|}\,\Omega_{E/K}\, R_{E/K}} = \frac{\# Ш(E/K)}{(\# E_{tor}(K))^2} \cdot \prod c_\ell$$

$z_K \in E(K)$ generator

$$R_{E/K} = \langle z_K, z_K \rangle_{NT} = \frac{1}{m^2} \langle y_K, y_K \rangle_{NT}$$

$y_K = m \cdot z_K$

so expect

$$m^2 = \frac{\#Ш(E/K)}{\#E(K)^2}\cdot\prod c_\ell$$

---

p splits in K, $p \nmid N_E$

$$\begin{tikzcd}
K_\infty \arrow[d, dash, "\mathbb{Z}_p^2"'] \arrow[dr, dash] & \\
 & K_\infty^{ac} \arrow[dl, dash, "\mathbb{Z}_p = \Gamma_{ac}"] \\
K \arrow[d, dash] & \\
\mathbb{Q} &
\end{tikzcd}$$

anticyclotomic $\mathbb{Z}_p$-ext'n of K

$p = v\bar{v}$
$\qquad \text{in } K$

$L(E, \chi, 1)$

$\chi : G_K \twoheadrightarrow \Gamma_{ac} \longrightarrow \bar{\mathbb{Q}}_p^\times$

∞ - order

HT	v	$\bar{v}$
	n	$-n$

$n > 1$ $\quad$ To $\quad$ $T(\chi)$

$1+n$	$1-n$
n	$-n$
> 0	≤ 0

$$\frac{L(E, \chi, 1)}{\Omega_K^{4n}}$$

There exists a p-adic
L-function

$$L_{BDP}(E/K) \in \mathbb{Z}_p^{ur}[\![\Gamma_{ac}]\!]$$

$$\phi_\chi(L_{BDP}) = (*)\,\Omega_p^{4n} \frac{L(E,\chi,1)}{\Omega_K^{4n}}$$

for χ as on preceding..

$$\phi_{1}(L_{BDP}) = \left(\frac{1-a_p(E)+p}{p} \cdot \log_{E(K_p)} y_K\right)^2$$

$$\underset{m}{\mathbb{Z}^{\cancel{k}}} \subseteq E(K) \subseteq \underset{\mathbb{Q}_p}{\overset{}{E(\underset{\shortparallel}{K_v})}} \xrightarrow{\log} K_v$$

$$\log y_K = \cancel{\mathbb{Z}_p \cdot y_K}$$

$$\left[E(K_v)/_{tor} : \mathbb{Z}_p \cdot y_K \right]$$

$$= m_p$$

$$m_p = \begin{pmatrix} p\text{-part} \\ \text{of } m \end{pmatrix} \cdot \left[E(K_v) : \mathbb{Z}_p \cdot E(K) \right]^{\cancel{tor}}$$

$$m_{loc}$$

An anticyclotomic MC

$$M = T \otimes_{\mathbb{Z}_p} \Lambda_{ac}^{*} \circlearrowleft \rho \otimes \Psi_{ac}^{-1}$$

$$\Lambda_{ac} = \mathbb{Z}_p[\![\Gamma_{ac}]\!]$$

$$\Psi_{ac} : G_K \twoheadrightarrow \Gamma_{ac} \hookrightarrow \Lambda_{ac}^{\times}$$

$$S_{BDP} = \left\{ c \in H^1(K, M) \;\middle|\; \begin{array}{l} \mathrm{res}_w c = 0 \quad \forall w \nmid p \\ \mathrm{res}_v c \text{ anything} \\ \mathrm{res}_{\bar{v}} c = 0 \end{array} \right\}$$

$p = v\bar{v}$

$$X_{BDP} = S^{*}_{BDP}$$

<u>Main Conjecture</u>:

(i) X_{BDP} is a torsion Λ_{ac}-module

(ii) $\xi(X_{BDP})$ is generated by the L_{BDP} in $\Lambda_{ac}^{ur} = \mathbb{Z}_p^{ur}[[\Gamma_{ac}]]$.

Control thm $\quad \gamma \in \Gamma_{ac}$ top'l gen

$$\to \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(E) \to S_{BDP}[\gamma-1] \to \prod_{w \mid N_E} H^1_{ur}(K_w, W) \times (P)$$

surjective if $\mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(E)$ is finite

So (as before):

$$\# \, X_{BDP} / (\gamma-1) X_{BDP} = \# \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(E)$$

$\times$ tamagawa factors

$\times$ term at $v \; \bar{v}$

Consequence of MC

$$\# \cancel{X}^{BDP}_{r-1} = \# \mathrm{Sel}_{rel,str}(E) \times \prod_{w \mid NE} c_w \times (v, \bar{v})$$

$$\| $$

$$\# \Lambda_{ac} / (g, r-1)$$

$$(g) = \xi(X_{BDP})$$

$$\| $$

$$\# \mathbb{Z}_p \Big/ \Big(\frac{1-a_p+p}{p} \underbrace{\log y_K}_{\text{index of}}\Big)^2$$

m: p-part $\dfrac{E(K) \text{ in } E(K_v)}{M_{loc}}$

p ord

$E[p]$ irreducible G_K-rep

$$
\begin{tikzcd}
0 \arrow[d] \\
\mathrm{Sel}_{\mathrm{ord},\mathrm{rel}}(T) \arrow[d] \\
H^1(K_{\bar v},T)/\mathrm{im}\, H^1(K_{\bar v},T^+) \simeq \mathbb{Z}_p \arrow[d, "\leftarrow \text{not zero}"] \\
X_{\mathrm{ord},\mathrm{rel}} = \mathrm{Sel}_{p^\infty}(E/K)^\vee \simeq \mathbb{Z}_p \oplus \underbrace{M \oplus M}_{\# = \# Ш(E/K)[p^\infty]} \arrow[d] \\
X_{\mathrm{ord},\mathrm{str}} \arrow[d] \\
0
\end{tikzcd}
$$

$\# X_{\mathrm{ord},\mathrm{str}} = m_{\mathrm{loc}} \cdot \# Ш(E/K)[p^\infty]$

$$S_{ord,rel}(T) = S_{ord,ord}(T)$$

$$\mathrm{Sel}_{ord,rel}(T) = S_{ord,ord}(T) = \mathbb{Z}_p$$

coker rel $\longrightarrow$

$h \geq$ nth $= m_{loc}$

$$\begin{tikzcd}
\mathrm{Sel}_{ord,rel}(T) \arrow[d] \\
\operatorname{im} H^1(K_v, T^*) \cong \mathbb{Z}_p \arrow[d] \\
X_{rel,str} = \mathrm{Sel}_{rel,str}(E)^* \arrow[d] \\
X_{ord,str} \overset{\#}{=} m_{loc} \cdot \# Ш(E/K)[p^\infty] \arrow[d] \\
0
\end{tikzcd}$$

$$m_{loc}^2 \cdot \# Ш(E/K)[p^\infty].$$

$$\# \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(E) \cdot \prod_{w \mid N_E} c_w$$

$$\|$$

$$m_{\mathrm{loc}}^2 \cdot \# Ш(E/K)[p^\infty] \cdot \prod_{w \mid N_E} c_w$$

$$\|$$

$$\# \; \mathbb{Z}_p \Big/ \Big(\frac{1-a_p+p}{p} \underbrace{\log y_K}_{m_{\mathrm{loc}} \cdot m_{p\text{-part}}} \Big)^2$$

$$\begin{pmatrix} p\text{-part} \\ \text{of } m \end{pmatrix}^2 = \# Ш(E/K)[p^\infty] \cdot \prod_{w \mid N_E} c_w$$

We even have progress toward the BDP IMC

essentially:

$$L_{BDP} \mid \xi(X_{BDP})$$

mostly due to X. Wan

THEOREM (Jetchev – S. – Wan)

Let $E/\mathbb{Q}$ be an elliptic curve with $\mathrm{ord}_{s=1} L(E,s) = 1$.

Suppose E is semistable.

($N_E = \square$-free)

Then

$$\left| \frac{L'(E,1)}{\Omega_E \, R_{E/\mathbb{Q}}} \right|_p^{-1} = \left| \begin{matrix} \text{expected} \\ \text{value} \end{matrix} \right|_p^{-1}$$

if $E[p]$ is an irreducible $G_{\mathbb{Q}}$-rep and $p \nmid N_E$, $p \geq 3$.