<u>THEOREM</u> Let $E/\mathbb{Q}$ be an elliptic curve. Let p be a prime s.t.

(a) $p \geq 3$

(b) E has (good) ordinary reduction at p

(c) $E[p]$ is an irred. $G_{\mathbb{Q}}$-rep

(d) there exists a prime $\ell \| N_E$ s.t. $E[p]$ is ramified at ℓ.

Then ($X(E/\mathbb{Q}_\infty)$ is Λ-torsion)

$$\xi(X(E/\mathbb{Q}_\infty)) = (\mathcal{L}_p(E/\mathbb{Q}_\infty))$$

That is, the Iwasawa MC holds for E (and p).

Kato: $\xi(X(E/\mathbb{Q}_\infty)) \mid (\mathcal{L}_p(E/\mathbb{Q}_\infty))$

S.-Urban:

$\cancel{\xi(X}$

$(\mathcal{L}_p(E/\mathbb{Q}_\infty)\,\mathcal{L}_p(E^K/\mathbb{Q}_\infty)) \mid \xi(E/\mathbb{Q}_\infty)\,\xi(E^K/\mathbb{Q}_\infty)$

K aux. imag. quad. field

E^K K-twist

together: $=$

Kato: $\mathbb{T} = T \otimes_{\mathbb{Z}_p} \Lambda \qquad \mathbb{T}^+ = T^+ \otimes_{\mathbb{Z}_p} \Lambda$

$$\begin{tikzcd}
\mathrm{Sel}_{\mathrm{rel}}(\mathbb{T}) \arrow[d, "\mathrm{res}_p"] & \supseteq & Z_{\mathrm{Kato}} \arrow[d, mapsto] \\
\dfrac{H^1(\mathbb{Q}_p, \mathbb{T})}{\mathrm{im}\, H^1(\mathbb{Q}_p, \mathbb{T}^+)} \arrow[d, "\mathrm{res}_p'"] & \xrightarrow[\sim]{\mathrm{Col}} & \Lambda \\
X = S^* \arrow[d] & & \\
X_{\mathrm{str}} \arrow[d] & & \\
0 & &
\end{tikzcd}$$

free Λ-mod of rk 1.

$\mathrm{Col}(Z_p)$

Euler system machine

$\Downarrow$

$$\xi(X_{\mathrm{str}}) \;\Big|\; \xi\left(\frac{\mathrm{Sel}_{\mathrm{rel}}(\mathbb{T})}{Z_{\mathrm{Kato}}}\right)$$

$$0$$
$$\downarrow$$
$$\frac{\mathrm{Sel}_{\mathrm{rel}}(T)}{Z_{\mathrm{Kato}}} \qquad \Big) \text{ torsion}$$
$$\downarrow \mathrm{res}_p$$
$$\frac{H^1(\mathbb{Q}_p, T)}{\mathrm{im}\, H^1(\mathbb{Q}_p, T^+),\ \mathrm{res}_p Z_{\mathrm{Kato}}} \simeq \Lambda/(\mathcal{L}_p) \qquad \text{torsion}$$
$$\downarrow$$
$$X_0 \qquad \text{torsion } \Lambda\text{-module}$$
$$\downarrow$$
$$X_{\mathrm{str}} \qquad \text{torsion}$$
$$\downarrow$$
$$0$$

$$\xi\left(\frac{\mathrm{Sel}_{\mathrm{rel}}(\pi)}{Z_{\mathrm{kato}}}\right)\underbrace{\xi(X)} = \underbrace{\xi(X_{\mathrm{str}})}\,\xi\left(\frac{\Lambda}{(I_p)}\right)$$

divides.

divides

$$\xi(X) \mid \xi\left(\frac{\Lambda}{(I_p)}\right) = (I_p)$$

What about the other divisibility?

$H^1(\mathbb{Q}, V)$ classifies $G_{\mathbb{Q}}$-reps:

$$0 \to V \longrightarrow X \longrightarrow \mathbb{Q}_p \longrightarrow 0$$

$$x \longmapsto 1$$

$$\sigma(x) - x \in V$$

strategy:

Eisenstein series with constant term on L-value of interest.

$\equiv$ cuspform modulo constant term

use Galois rep. assoc. to cuspidal (eigenforms) rep's

$$GL_2 \supset \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = B$$

$$\chi_1, \chi_2$$

$$f : GL_2 \longrightarrow \mathbb{C}$$

$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \chi_1(a)\,\chi_2(d)\,f(g)$$

$$E(g) = \sum_{\gamma \in B(\mathbb{Q}) \backslash GL_2(\mathbb{Q})} f(\gamma g)$$

$GU(2,2)$ K imag.

$\cup$

$P = MN$

$$M = GU(1,1) \times \underbrace{\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m}_{=K^\times}$$

$$GU(1,1) = \frac{GL_{2/\mathbb{Q}} \times K^\times}{\mathbb{Q}^\times}$$

allow us to vary the cyclotomic dir

$f_E \longrightarrow (\pi, \psi, \chi)$

$\chi_\pi = \psi|_{\mathbb{A}_\mathbb{Q}^\times}$

$L(\pi, \chi, 1) \cdots$

$GU(2,2) \longrightarrow$ 4 dim'l reps of G_K

E is $\longrightarrow$ $\begin{pmatrix} \chi_1 & * & * \\ & \boxed{\begin{matrix} \chi_2 & * \\ & \rho_f \end{matrix}} & \end{pmatrix}$

(f, ψ, χ)

π

$$\begin{pmatrix} \chi & * & \\ & \varepsilon & * \\ & & \rho_E \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon & \bullet & * \\ & 1 & * \\ & & \rho_E \end{pmatrix}$$

Consequences

(1) $L(E,1) \neq 0$

$\Rightarrow \mathrm{rk}\, E(\mathbb{Q}) = 0$

$$\# X/(\gamma-1)X = \# \Lambda/(\mathcal{L}_p(E/\mathbb{Q}_\infty), \gamma-1)$$

$$= \# \mathbb{Z}_p/(1-\tfrac{1}{\alpha_p})^2 \frac{L(E,1)}{\Omega_E}$$

$\|$

$\# \mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$

$\cdot \prod_{\ell | N_E} c_\ell$

$\cdot \# \left(\mathbb{Z}_p/(\alpha_p - 1)\right)^2$

p-part BSD formula

$\Rightarrow$

$$\left| \# Ш(E/\mathbb{Q})[p^\infty] \cdot \prod_{\ell | N_E} c_\ell \right|_p^{-1}$$

$$= \left| \frac{L(E,1)}{\Omega_E} \right|_p^{-1}$$

BSD

(1) $\operatorname{ord}_{s=1} L(E,s) = \underset{\substack{\shortparallel \\ r}}{\operatorname{rk} E(\mathbb{Q})}$

(2) $$\frac{L^{(r)}(E,1)}{r!\,\Omega_E R_{E/\mathbb{Q}}} = \frac{\#Ш(E/\mathbb{Q})}{(\#E_{tor}(\mathbb{Q}))^2} \cdot \prod c_\ell$$

(2) $L(E,1) = 0$

$$\Rightarrow \# X/(\gamma-1)X = \infty$$

$$\Rightarrow \# \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = \infty$$

That is $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$ has corank ≥ 1

$$0 \to E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) \to Ш(E/\mathbb{Q})[p^\infty] \to 0$$

Remarks

(1) E has CM

IMC was proved by Rubin

(2) Kato's IMC w/out L-functions

holds also for E having

supersingular

reduction.

(3) Greenberg – Vatsal

prove IMC for cases

where $E[p]$ is reducible.

(4) criteria for establishing IMC

when the prime $\ell \| N_E$ doesn't

exist!