Recap: E ell. curve

N_E cond

$p \nmid N_E$ ordinary red.

$T = T_p E \quad 0 \to T^+ \to T \to T^- \to 0$

$W = E[p^\infty]$ $\quad G_{\mathbb{Q}_p}$-filtration

$= T \otimes \mathbb{Q}_p/\mathbb{Z}_p$

$\chi : G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_p^\times \quad \mathcal{O}_\chi$

$W(\chi) = W \underset{\mathbb{Z}_p}{\otimes} \mathcal{O}_\chi \circlearrowleft \rho_{E,p} \otimes \chi$

$Sel(E,\chi) \subseteq H^1(\mathbb{Q}, W(\chi))$

$res_\ell\, c = 0 \quad \ell \neq p$

$res_p\, c \in im(W^+(\chi))_{d}$

$$\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \simeq \mathbb{Z}_p$$

$$\begin{array}{ccc} \Gamma & & \mathbb{Z}_p \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \gamma & \longmapsto & 1 \end{array}$$

$$\Lambda = \mathbb{Z}_p[\![\Gamma]\!] \simeq \mathbb{Z}_p[\![T]\!]$$

$$\gamma \longmapsto 1+T$$

$$\Lambda^* = \mathrm{Hom}_{\mathrm{cts}}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$$

$$\Psi : G_{\mathbb{Q}} \twoheadrightarrow \Gamma \hookrightarrow \Lambda^\times$$

$$M = T \otimes_{\mathbb{Z}_p} \Lambda^* \circlearrowleft \rho_{E,p} \otimes \Psi^{-1}$$

$$(M \otimes_{\mathbb{Z}_p} \mathcal{O}_\chi)[\gamma - \chi(\gamma)] = W(\chi^{-1})$$

$$S = S(E/\mathbb{Q}_\infty) \subseteq H^1(\mathbb{Q}, M)$$

$$\mathrm{res}_\ell\, c = 0 \quad \forall\, \ell \neq p$$

$$\mathrm{res}_p\, c \longmapsto 0 \text{ in } H^1(I_p, M^-)$$

$$E[p]^{G_{\mathbb{Q}}} = 0$$

$$\downarrow$$

$$\mathrm{Sel}(E, \bar{\chi}^{-1}) \underset{\text{finit index}}{\subseteq} \left(S \otimes_{\mathbb{Z}_p} \mathcal{O}_\chi\right)[\gamma - \chi(\gamma)]$$

$$\#\left(\mathbb{Z}_p/a_p-1\right)^2$$

$$\chi = 1$$

$$\hookrightarrow \prod_{\ell \mid N_E} H^1_{ur}(\mathbb{Q}_\ell, W(\bar{\chi}^{-1})) \times (\qquad)_p$$

$$X = X(E/\mathbb{Q}_\infty) = S^*$$

finitely-gen. Λ-module

If torsion, then

$$\xi(X) = \text{char ideal} \subseteq \Lambda$$

$$= (g) \qquad \frac{L(E, \chi^{-1}, 1)}{\Omega_E}$$

$$\# \; X \otimes_{\mathbb{Z}_p} \mathcal{O}_\chi \Big/ (\gamma - \chi(\gamma)) \cdot X \otimes \mathcal{O}_\chi$$

" "

$$= \# \, \mathrm{Sel}(E, \chi^{-1}) \cdot \prod_{\ell \mid N_E} \# H^1_{ur}(\mathbb{Q}_\ell, W(\chi^{-1}))$$

$$\times \left(\quad \right)_p$$

$\# \cancel{X}$

$$\# \frac{X \otimes_{\mathbb{Z}_p} \mathcal{O}_\chi}{(\gamma - \chi(\gamma))\, X \otimes_{\mathbb{Z}_p} \mathcal{O}_\chi} = \# \frac{\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_\chi}{(\gamma - \chi(\gamma), g)}$$

$$= \# \frac{\mathcal{O}_\chi [[T]]}{T - (\chi(\gamma) - 1),\ g(T)}$$

$$= \# \, \mathcal{O}_\chi / g(\chi(\gamma) - 1)$$

$$g(\chi(\gamma) - 1) \overset{?}{=} \text{``}\, \frac{L(E, \chi^{-1}, 1)}{\Omega_E} \,\text{''}$$

p-adic L-function of E

$$E \longleftrightarrow f_E = \sum_{n=1}^{\infty} a_n q^n$$

(Wiles, et al) $\in S_2(\Gamma_0(N_E))$

$$L(E,s) = L(f_E, s)$$

$$L(E,\chi,s) = L(f_E,\chi,s)$$

$$f_E \longrightarrow \Omega_{f_E}^{\pm}$$

$$\frac{L(f_E,\chi,1)}{\Omega_{f_E}^{+}}$$ χ even

is algebraic

There exists

$$\mathcal{L} = L_p(E/\mathbb{Q}_\infty) \in \Lambda$$

s.t. if $\chi : \Gamma \longrightarrow \bar{\mathbb{Q}}_p^\times$
finite order

$$\left[\phi_\chi : \underset{\text{hom.}}{\Lambda \longrightarrow \mathbb{Q}_p}, \quad \phi_\chi|_\Gamma = \chi \right]$$

$$\phi_\chi(\mathcal{L}) = e_p(\chi) \frac{L(f_E, \chi^{-1}, 1)}{\Omega^+_{f_E}}$$

$\chi(\gamma) = \zeta_{p^t}$

$$e_p(\chi) = \begin{cases} \alpha_p^{-1}\left(1 - \frac{1}{\alpha_p}\right)^2 & \chi = \mathbb{1} \\ \alpha_p^{-(t+1)} \frac{p^{t+1}}{G(\chi^{-1})} & \chi \neq \mathbb{1} \end{cases}$$

2 - constructions :

1) Modular symbols

$$\text{“} \int_0^{i\infty} f(iy)\, y^{s} \frac{dy}{y} = L(f,s) \text{”}$$

2) Rankin - Selberg

$$\langle f, E_1, E_2 \rangle$$

unit

$$= \frac{L(f,\chi_1,1)}{\Omega_f^{+}} \frac{L(f,\chi_2,1)}{\Omega_f}$$

The Main Conjecture for $E/\mathbb{Q}_\infty$ (with L-function)

(i) $X = X(E/\mathbb{Q}_\infty)$ is a torsion Λ-module

(ii) $\xi(X) = (\mathcal{L})$

in $\Lambda \underset{\mathbb{Z}_p}{\otimes} \mathbb{Q}_p$

and even in Λ

if $\cancel{E[p] \cong_{G_{\mathbb{Q}}} \mathbb{Z}/\mathbb{Q}}$

$E[p]$ is an irreducible

$G_{\mathbb{Q}}$-rep.

The MC w/out L-functions

$$\mathbb{T} = T \underset{\mathbb{Z}_p}{\otimes} \Lambda \circlearrowleft \rho_{E,p} \otimes \Psi \qquad \mathbb{T}^+ = T^+ \otimes \Lambda$$

$$\begin{array}{ccccc}
 & & \mathrm{Sel}_{\mathrm{rel}}(\mathbb{T}) & & \\
 & & \downarrow{\scriptstyle \mathrm{res}_p} & & \\
\Lambda & \overset{\mathrm{Col}}{\xleftarrow{\sim}} & H^1(\mathbb{Q}_p,\mathbb{T}) / H^1(\mathbb{Q}_p,\mathbb{T}^+) & \simeq & H^1(\mathbb{Q}_p, M^-)^* \\
 & & \downarrow{\scriptstyle \mathrm{res}_p^*} & & \\
 & & X & = & S^* \\
 & & \downarrow & & \\
 & & X_{\mathrm{str}} & = & S_{\mathrm{str}}^* \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}$$

$$\text{ker}\ \text{im}\left\{ \begin{array}{c} H^1(\mathbb{Q}_p, M) \\ \downarrow \\ H^1(\mathbb{Q}_p, M^-) \end{array} \right\}^*$$

Kato constructs

$$z_{\mathrm{Kato}} \subseteq \mathrm{Sel}_{\mathrm{rel}}(\mathbb{T})$$

Λ-module

free, rank 1

$$\mathrm{Col}(\mathrm{res}_p(z_{\mathrm{Kato}})) = (\mathcal{L})$$

MC w/out L-function

(i) $Sel_{rel}(T)$ is a rank 1 Λ-module

(ii) $$\xi\left(\frac{Sel_{rel}(T)}{Z_{Kato}}\right) = \xi(X_{str})$$

The two main conjectures are equivalent