E elliptic curve $/\mathbb{Q}$

N_E conductor

$p \nmid N_E$ ordinary reduction

$T = T_p E \qquad 0 \to T^+ \to T \to T^- \to 0$

$G_{\mathbb{Q}_p}$ filtration

$V = T \underset{\mathbb{Z}_p}{\otimes} \mathbb{Q}_p \qquad W = V/T = T \otimes \mathbb{Q}_p/\mathbb{Z}_p$

$= E[p^\infty]$

$\chi : G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ finite order

$\mathcal{O}_\chi$ ring of integers of the finite ext'n of $\mathbb{Q}_p$ gen by χ

$T(\chi) = T \underset{\mathbb{Z}_p}{\otimes} \mathcal{O}_\chi \circlearrowleft \rho_{E,p} \otimes \chi$

$G_{\mathbb{Q}}$-action

$V(\chi)$, $W(\chi)$ . . .

$\mathrm{Sel}(E,\chi)$

$$= \quad c \in H^1(\mathbb{Q}, W(\chi))$$

$$\mathrm{res}_\ell\, c = 0 \quad \forall\, \ell \neq p$$

$$\mathrm{res}_p\, c \in \mathrm{im}\left\{ \begin{array}{c} H^1(\mathbb{Q}_p, W^+(\chi)) \\ \downarrow \\ H^1(\mathbb{Q}_p, W(\chi)) \end{array} \right\}_{\mathrm{div}}$$

$$\| $$

$$\in \ker \left\{ \begin{array}{c} H^1(\mathbb{Q}_p, W(\chi)) \\ \downarrow \\ H^1(\mathbb{Q}_p, W^-(\chi)) \end{array} \right\}_{\mathrm{div}}$$

$\mathbb{Q}_\infty / \mathbb{Q}$ cyclotomic $\mathbb{Z}_p$-ext'n

$\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty / \mathbb{Q}) \simeq \mathbb{Z}_p$

$\Lambda = \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T]]$

$\Psi : G_{\mathbb{Q}} \twoheadrightarrow \Gamma \hookrightarrow \Lambda^*$

$\Lambda^* = \mathrm{Hom}_{ct}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$

$\circlearrowright$

$G_{\mathbb{Q}}$ acts via Ψ^{-1}

$M = T \otimes_{\mathbb{Z}_p} \Lambda^* \circlearrowleft \rho_{E,p} \otimes \Psi^{-1}$

$\chi : \Gamma \to \bar{\mathbb{Q}}_p^\times$ finite order

$(M \otimes_{\mathbb{Z}_p} \mathcal{O}_\chi)[\gamma - \chi(\gamma)] = W(\chi^{-1})$

$\gamma \in \Gamma$ top'l generator

$$M = T \otimes_{\mathbb{Z}_p} \Lambda^{\iota} = \mathrm{Hom}_{\mathrm{cts}}(\Lambda, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{W}$$

$$M \otimes_{\mathbb{Z}_p} \mathcal{O}_\chi = \mathrm{Hom}_{\mathrm{cts}, \mathcal{O}_\chi\text{-lin}}\left(\frac{\Lambda \otimes \mathcal{O}_\chi}{(\gamma - \chi(\gamma))}, T \otimes \mathcal{O}_\chi \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)$$

$[\gamma - \chi(\gamma)]$ $\qquad \mathcal{O}_\chi(\chi^{-1})$

<u>exercise</u>

$$E[p] \simeq M[r-1, p]$$

$$M[r-1, p]^{G_{\mathbb{Q}}} = 0$$

$$M[r-1, p]^{*} = M^{*} / (r-1, p) M^{*}$$

$$
\begin{tikzcd}
 & H^1(\mathbb{Q}, W) \arrow[r, "\sim"] \arrow[d] & H^1(\mathbb{Q}, M)[\gamma-1] \arrow[d] \\
{} \arrow[r] & H^1(I_p, W^-)^{G_{\mathbb{Q}_p}} \arrow[r, two heads] & H^1(I_p, M^-)[\gamma-1]^{G_{\mathbb{Q}_p}}
\end{tikzcd}
$$

$$\left((M^-)^{I_p} \big/ (\gamma-1) M^{-\,I_p} \right)^{G_{\mathbb{Q}_p}} = (W^-)^{G_{\mathbb{Q}_p}}$$

$$\cancel{= W^-[\mathrm{Frob}-}$$

$$= W^-[\alpha_p - 1]$$

$$\uparrow$$

$$0$$

$$(M^-)^{I_p} = M^-[\gamma-1] = W^-$$

has order $\# \mathbb{Z}_p / \alpha_p - 1$

$$E[p]^{G_{\mathbb{Q}}} = 0$$

$$\mathrm{Sel}(E, \bar{\chi}^{-1}) \subseteq \left(S \otimes_{\mathbb{Z}_p} \mathcal{O}_\chi\right)[\gamma - \chi(\gamma)]$$

finite index

$$\chi = \mathbb{1}$$

$$0 \to \mathrm{Sel}(E) \to S[\gamma - 1] \to \prod_{\ell \mid N_E} \overbrace{H^1_{ur}(\mathbb{Q}_\ell, W)}^{p\text{-part of } c_\ell(E)}$$

$$\times \left(\ \sim \ \right)$$

$$\# \left(\mathbb{Z}_p / a_p - 1 \right)^2$$

If $\uparrow$ finite

then $\to$ is a surjection.

$$(\mathbb{Q}, M) \xrightarrow{\gamma-1} H^1(\mathbb{Q}, M)$$

$${}^1(\mathbb{Q}, M)[\gamma-1]$$

$$\begin{tikzcd} \mathbb{Q}_\infty \arrow[d, dash, "\Gamma_n"] \arrow[dd, dash, bend right] \\ \mathbb{Q}_n \arrow[d, dash, "\mathbb{Z}/p^n"] \\ \mathbb{Q} \end{tikzcd}$$

$\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$

$\curvearrowright$

$$S = \varinjlim \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_n)$$

$H^1(\mathbb{Q}_n, W)$

$H^1(\mathbb{Q}, \mathrm{Ind}(W))$

Iwasawa Selmer group

$$S = S(E/\mathbb{Q}_\infty)$$

$$\left\{ c \in H^1(\mathbb{Q}, M) : \begin{array}{l} \ell \neq p \quad \mathrm{res}_\ell\, c = 0 \ (\text{unramified}) \\ \ell = p \quad \mathrm{res}_p\, c \in \cancel{\mathrm{im}\ H^1(} \ \ker \left(\begin{array}{c} H^1(\mathbb{Q}_p, M) \\ \downarrow \\ H^1(I_p, \bar{T} \otimes \Lambda^*) \end{array} \right) \end{array} \right\}$$

$$M[\gamma - \chi(\gamma)] = W(\chi^{-1})$$

$$S[\gamma - \chi(\gamma)] = \ ?$$

$$Sel(E, \chi^{-1})$$

$$E[p]^{G_{\mathbb{Q}}} = 0 \quad \underline{\text{Assumption}}$$

$$0 \to W \to M \xrightarrow{\gamma - 1} M \to 0$$

$$M^{G_{\mathbb{Q}}} \to H^1(\mathbb{Q}, W) \to H^1$$

$$\| \qquad\qquad H^1(\mathbb{Q}, W) \xrightarrow{\sim} H$$

$$0$$

S is a cofinitely-generated
Λ-module

$X = X(E/\mathbb{Q}_\infty) = S^*$

is finitely generated.

$H^1(G_\Sigma, M)[p, \gamma - 1]$

$\Sigma = \{\ell \mid p N_E \cdot \infty\}$

$H^1(G_\Sigma, M)[\gamma - 1] = H^1(G_\Sigma, W)$

$H^1(G_\Sigma, W)[p] = H^1(G_\Sigma, E[p])$

$$X \sim \prod_{i=1}^{r} \Lambda/(f_i)$$

$$\xi(X) = (f_1 \cdots f_r) \subseteq \Lambda$$

<u>if</u> X torsion

<u>Suppose</u>

$$0 \to X \hookrightarrow \prod \Lambda/(f_i) \to \underset{\text{finite}}{\overset{\mathfrak{n}}{\text{module}}} \to 0$$

multiply by T $(= \gamma - 1)$

$$0 \to \mathfrak{n}[T] \to X/TX \to \prod \Lambda/(f_i, T) \to \mathfrak{n}/T \to 0$$

$\nwarrow$ if $f_i(0) \neq 0$, $i = 1, \ldots r$

$$\prod \Lambda/(f_i, T) = \mathbb{Z}_p/f_i(0)$$

$$\# X/TX = \# \mathbb{Z}_p/\prod f_i(0)$$

$T =$

$$= \# \mathbb{Z}_p/g(0)$$

$$\xi(X) = (g) \subseteq \Lambda$$

So if $X = X(E/\mathbb{Q}_\infty)$ is Λ-torsion & has no pseudo null submodule, then

$$\# X/(\gamma-1)X = \# S[\gamma-1] = \# \mathrm{Sel}(E) \times \prod c_\ell \times \#\left(\mathbb{Z}_p/a_p-1\right)^2$$

$$\# \Lambda/(g,T) = \# \mathbb{Z}_p/g(0)$$

Exercise:

- $\mathrm{Sel}(E/\mathbb{Q}_\infty)$ is finite then X is Λ-torsion (X/TX has finite order)

(!) $\mathrm{Sel}(E/\mathbb{Q}_\infty)$ is finite, then X has no pseudonull submodule.