E elliptic curve $/\mathbb{Q}$

$E(\mathbb{Q})$ f.g. ab. group

<u>BSD</u>

(i) $\mathrm{rk}\, E(\mathbb{Q}) \overset{?}{=} \mathrm{ord}_{s=1} L(E,s)$

(ii) $r = \mathrm{rk}\, E(\mathbb{Q})$

$$\frac{L^{(r)}(E,1)}{r!\,\Omega_E\, R_{E/\mathbb{Q}}} \overset{?}{=} \frac{\#\text{Ш}(E/\mathbb{Q})}{(\#E_{\mathrm{tor}}(\mathbb{Q}))^2} \times \prod_{\ell \mid N_E} c_\ell$$

$E(\mathbb{Q}) / mE(\mathbb{Q})$ is finite

$m \geq 1$

$P \in E(\mathbb{Q})$

$Q \in E(\overline{\mathbb{Q}})$ s.t. $mQ = P$

$\sigma \in G_{\mathbb{Q}}$

$\phi(\sigma) = \sigma Q - Q \in E[m]$

1-cocycle

$c_P = [\phi] \in H^1(\mathbb{Q}, E[m])$

$\ell \nmid m N_E$

$\phi(\sigma)$ is unramified at ℓ

$\Sigma = \{\ell \mid m N_E \infty\}$

G_Σ

$$c_P \in \underbrace{H^1(G_\Sigma, E[m])}_{\text{finite}}$$

Selmer group $\mathrm{Sel}_m(E/\mathbb{Q})$

$$\begin{tikzcd}
E(\mathbb{Q})/mE(\mathbb{Q}) \arrow[r, hook] \arrow[d] & H^1(\mathbb{Q}, E[m]) \arrow[d, "\mathrm{res}_\ell"] \\
E(\mathbb{Q}_\ell)/mE(\mathbb{Q}_\ell) \arrow[r, hook, "\kappa_\ell"] & H^1(\mathbb{Q}_\ell, E[m])
\end{tikzcd}$$

$$\mathrm{Sel}_m(E/\mathbb{Q}) = \left\{ \begin{array}{l} c \in H^1(\mathbb{Q}, E[m]) \text{ s.t.} \\ \quad \mathrm{res}_\ell\, c \in \mathrm{im}\,\kappa_\ell \ \forall \ell, \infty \end{array} \right\}$$

$\operatorname{im} \kappa_\ell$ = unramified at ℓ classes

$\ell \nmid m N_E$

$$\mathrm{Sel}_m(E/\mathbb{Q}) \subseteq H^1(G_\Sigma, E[m])$$

$$
\begin{tikzcd}
0 \arrow[r] & E(\mathbb{Q})/mE(\mathbb{Q}) \arrow[r] & \mathrm{Sel}_m(E/\mathbb{Q}) \arrow[d] \\
& & Ш(E/\mathbb{Q})[m] \arrow[d] \\
& & 0
\end{tikzcd}
$$

$Ш(E/\mathbb{Q})$

$\|$

$$\ker\left\{H^1(\mathbb{Q}, E) \longrightarrow \prod_{\ell \leq \infty} H^1(\mathbb{Q}_\ell, E)\right\}$$

$$0 \to E[m] \longrightarrow E \xrightarrow{m} E \to 0$$

take cohomology ...

$$E[m] \longrightarrow E[mm']$$

$$\mathrm{Sel}_m \longrightarrow \mathrm{Sel}_{mm'}$$

$$m = p^n$$

$$\varinjlim_n \mathrm{Sel}_{p^n}(E/\mathbb{Q}) = \mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$$

$$\varinjlim_n E(\mathbb{Q})/p^n E(\mathbb{Q}) = E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \simeq \left(\mathbb{Q}_p/\mathbb{Z}_p\right)^r$$

$$r = \mathrm{rk}\, E(\mathbb{Q})$$

$$\begin{tikzcd} 0 \arrow[r] & E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] & \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) \arrow[d] \\ & & Ш(E/\mathbb{Q})[p^\infty] \arrow[d] \\ & & 0 \end{tikzcd}$$

exact

expectations

- $Ш(E/\mathbb{Q})[p^\infty]$: finite
- if so then

$$Ш(E/\mathbb{Q})[p^\infty] \simeq N \oplus N$$

(by Cassels-Tate)

$m = p$

$$0 \to E(\mathbb{Q}) \otimes \mathbb{Z}/p\mathbb{Z} \to \underbrace{Sel_p(E/\mathbb{Q})}_{\substack{\text{suppose} \\ \simeq \mathbb{F}_p}} \to Ш(E/\mathbb{Q})[p] \to 0$$

expect:

either $\operatorname{rk} E(\mathbb{Q}) = 0$ but $E[p](\mathbb{Q}) \neq 0$

or

$\operatorname{rk} E(\mathbb{Q}) = 1$ & $E[p](\mathbb{Q}) = 0$

Suppose $E[p](\mathbb{Q}) = 0$

Can we prove

$$Sel_p(E/\mathbb{Q}) \cong \mathbb{F}_p \overset{?}{\Longrightarrow} \operatorname{rk} E(\mathbb{Q}) = 1$$

$m = p^\infty$

$\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$ has corank $= 1$

$$\Downarrow ?$$

$\mathrm{rk}\, E(\mathbb{Q}) = 1.$

$+$

$$\mathrm{ord}_{s=1} L(E,s) \stackrel{?}{=} 1$$

$Sel_{p^\infty}(E/\mathbb{Q}) \subseteq H^1(\mathbb{Q}, E[p^\infty])$

$E(\mathbb{Q}_\ell) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 0$

$\ell \neq p$

<u>$\ell = p$</u> $\quad p \nmid N_E$

ordinary, s.s. (2 poss.)

$T = T_p E = \varprojlim_n E[p^n]$

$T \otimes \mathbb{Q}_p/\mathbb{Z}_p = E[p^\infty]$

p <u>ordinary</u> $\quad G_{\mathbb{Q}_p}$ - filtration

$$0 \to T^+ \to T \to T^- \to 0$$

$$\begin{matrix} \wr\wr & \wr\wr & \wr\wr \\ \mathbb{Z}_p & \mathbb{Z}_p^2 & \mathbb{Z}_p \end{matrix}$$

$\circlearrowleft$ $\qquad\qquad\qquad\qquad$ $\circlearrowleft$

$G_{\mathbb{Q}_p}$ acts $\qquad\qquad$ $G_{\mathbb{Q}_p}$ acts by

$\varepsilon_{cyc}\, \alpha^{-1}$ $\qquad\qquad\qquad$ α

$$\alpha : G_{\mathbb{Q}_p} \longrightarrow G_{\mathbb{Q}_p}/I_p \longrightarrow \mathbb{Z}_p^\times$$

$$\| \\ \langle \mathrm{Frob}_p \rangle$$

$\alpha(\mathrm{Frob}_p) = \alpha_p$ the unit root of

$$X^2 - a_p(E)X + p$$

$$\operatorname{im} \kappa_p = \operatorname{im} \left\{ \begin{array}{c} H^1(\mathbb{Q}_p, T^+ \otimes \mathbb{Q}_p/\mathbb{Z}_p) \\ \downarrow \\ H^1(\mathbb{Q}_p, \underbrace{T \otimes \mathbb{Q}_p/\mathbb{Z}_p}_{= E[p^\infty]}) \end{array} \right\}_{\text{div}}$$

so $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$

is defined solely in

terms of $E[p^\infty]$ (not T)

s.s. $V = T_p E \underset{\mathbb{Z}_p}{\otimes} \mathbb{Q}_p = T \underset{\mathbb{Z}_p}{\otimes} \mathbb{Q}_p$

$H^1_f(\mathbb{Q}_p, V)$ Bloch-Kato

$\downarrow$ im $=: H^1_f(\mathbb{Q}_p, E[p^\infty])$

$H^1(\mathbb{Q}_p, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)$

$H^1_f(\mathbb{Q}_p, E[p^\infty])$

$= \operatorname{im} \kappa_p$

(always)

$T(x)$ $\qquad \mathbb{Z}_p \quad \mathbb{Z}_p[x]$

$V(x)$ $\qquad V = T \otimes \mathbb{Q}_p$

$W(x)$ $\qquad W = E[p^\infty]$

$\qquad = T \otimes \mathbb{Q}_p/\mathbb{Z}_p$

$\mathrm{res}_\ell = 0 \quad \forall\, \ell \neq p$

$\mathrm{res}_p \in H^1_f(\mathbb{Q}_p, W(x))$

$$H^1(\mathbb{Q}, W(\chi^{-1})) \supseteq \mathrm{Sel}(W(\chi^{-1}))$$

$$\ni c \quad \text{s.t.}$$

$$\mathrm{res}_\ell\, c = 0 \quad \forall\, \ell \neq p$$

$$\mathrm{res}_p\, c \in \underbrace{H^1_f(\mathbb{Q}_p, W(\chi^{-1}))}_{\mathrm{im}\ H^1_f(\mathbb{Q}_p, V(\chi^{-1}))}$$