Modular symbols

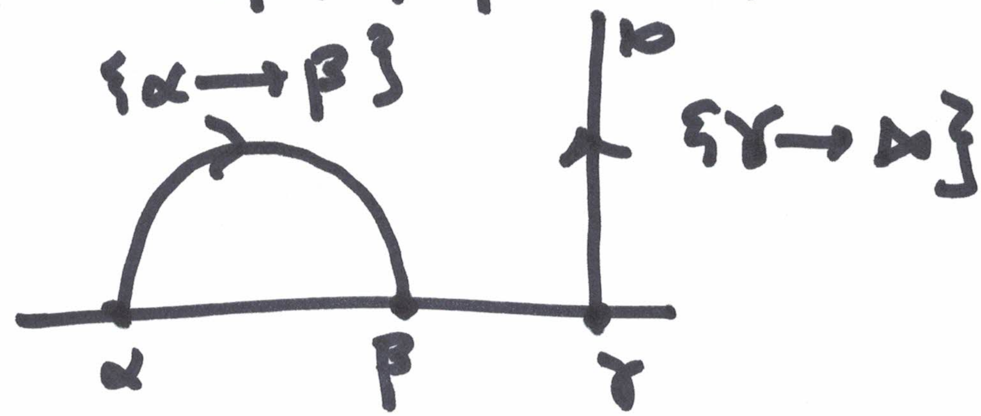
$X_1(p^r) = Y_1(p^r) \amalg C_1(p^r)$

$C_1(p^r) = \text{cusps} = \Gamma_1(p^r)\backslash\mathbb{P}^1(\mathbb{Q})$

$\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ modular symbol

$\{\alpha \to \beta\} \in H_1(X_1(p^r), C_1(p^r), \mathbb{Z})$

class of



Thm (Manin) $H_1(X_1(p^r), C_1(p^r), \mathbb{Z})$ is gen. by symbols $[u:v]_r$

$= \gamma\{0 \to \infty\} = \left\{\frac{b}{d} \to \frac{a}{c}\right\}$

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ $\quad (u,v) = (c,d) \bmod p^r\mathbb{Z}^2$

where $u, v \in \mathbb{Z}/p^r\mathbb{Z}$, $(u,v)=(1)$

w/ relations

$$[u:v]_r = [-u:-v]_r = -[-v:u]_r$$

$$= [u:u+v]_r + [u+v:v]_r.$$

$$[u:v]_r^+ := \frac{1}{2}\left([u:v]_r + [-u:v]_r\right)$$

$$\in H_1(X_1(p^r), C_1(p^r), \mathbb{Z}_p)^+$$

$$[u:v]_r^* = W_{p^r}[u:v]_r^+ \qquad W_{p^r} = \begin{pmatrix} 0 & -1 \\ p^r & 0 \end{pmatrix}$$

$$= \left\{ \frac{d}{bp^r} \longrightarrow \frac{c}{ap^r} \right\}^+.$$

$$C_1^0(p^r) = \left\{ \Gamma_1(p^r)\frac{a}{b} \;\middle|\; \begin{matrix}(a,b)=1 \\ p \nmid b\end{matrix} \right\}.$$

$$S_r^0 := H_1(X_1(p^r), C_1^0(p^r), \mathbb{Z}_p)^+$$

<u>Cor</u> S_r° is generated by the $[u:v]_r^*$ with $u, v \neq 0$ w/ relns. $[u:v]_r^* = [-u:v]_r^*$
$= -[v:u]_r^* = [u:u+v]_r^* + [u+v:v]_r^*$

$$S_r = H_1(X_1(p^r), \mathbb{Z}_p)^+ \subset S_r^\circ.$$

<u>Recall</u>: cup product pairing on cyclotomic p-units in $F_r = \mathbb{Q}(\mu_{p^r})$:

$$\mathcal{C}_r \times \mathcal{C}_r \xrightarrow{(\,,\,)_r} \underset{\substack{\| \\ Y_r}}{H^2(g_r, \mathbb{Z}_p(2))^+}$$

g_r = Galois group of max'l p-ram. extn. of F_r

Thm (Basuioc, S.)

$\exists\ \Pi_r : S_r^{\circ} \longrightarrow Y_r$,

$$\Pi_r([u:v]_r^*) = (1-\zeta_{p^r}^u, 1-\zeta_{p^r}^v)_r,$$

homom. of $\mathbb{Z}_p[\underbrace{(\mathbb{Z}/p\mathbb{Z})^\times}_{\Delta}]$

$j \in \Delta$ act on left by $\langle j \rangle^{-1}$

" " right by σ_j, $\chi_p(\sigma_j) = j$

Idea: $$\frac{1-\zeta^u}{1-\zeta^{u+v}} + \zeta^u \frac{1-\zeta^v}{1-\zeta^{u+v}} = 1$$

$$\Rightarrow \left(\frac{1-\zeta^u}{1-\zeta^{u+v}}, \frac{1-\zeta^v}{1-\zeta^{u+v}}\right)_r = 0.$$

$\Leftrightarrow$ last Manin relns. //

Thm (Fukaya–Kato)

$\Pi_r : S_r^{\circ} \longrightarrow Y_r$ factors through quotient by Eisenstein ideal

$I = (T_\ell - 1 - \ell\langle \ell \rangle$ ℓ prime $\neq p$,
$U_p - 1)$

Rmks: 1) $S_r^{\mathrm{ord}} = (S_r^{\circ})^{\mathrm{ord}}$

$\rightsquigarrow S_r / I S_r = S_r^{\circ} / I S_r^{\circ}$.

~~2d~~ $\rightsquigarrow \varpi_r : S_r / I S_r \longrightarrow Y_r$.

2) $[pu:v]^*_{r+1} \longmapsto [u:v]^*_r$

$(1-\zeta_{p^r}^{u}, 1-\zeta_{p^{r+1}}^{v})_{r+1}$

$\longmapsto (1-\zeta_{p^r}^{u}, 1-\zeta_{p^r}^{v})_r$

$\rightsquigarrow \varpi = \varprojlim \varpi_r : S/IS \longrightarrow Y.$

$$S = \varprojlim H_1(X_1(p^r), \mathbb{Z}_p)^+$$

$$Y = \varprojlim H^2(\mathfrak{g}_r, \mathbb{Z}_p(2))^+$$
$$= H^2_{Iw}(F_\infty, \mathbb{Z}_p(2))^+$$
$$= X_\infty(1)^+ = X_\infty^-(1).$$

<u>Sketch of proof</u>:

$\exists$ Siegel units $g_u \in \mathcal{O}(Y_1(p^r)/\mathbb{Z}[\frac{1}{p}])^\times \otimes_{\mathbb{Z}} \mathbb{Q}$
$u \in \mathbb{Z}/p^r\mathbb{Z} - \{0\}$.

$g_u \cup g_v \in H^2_{et}(Y_1(p^r)/\mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(2))$ $\overset{!!}{\ } H^2_{et}$

Beilinson–Kato elt.

1) $\exists$ Hecke-equivar. map

$z : S_r^0 \longrightarrow H^2_{et}$ $\quad z([u:v]_r^*)$
$= g_u \cup g_v$.

2) $\infty : \operatorname{Spec} \mathbb{Z}[\mu_{p^r}, \frac{1}{p}]^+ \hookrightarrow X_1(p^r)$

$\rightsquigarrow$ $\infty : H^2_{et} \longrightarrow H^2_{et}(\mathbb{Z}[\mu_{p^r}, \frac{1}{p}], \mathbb{Z}_p(2))^+$

$\cong Y_r$. Specialization at ∞

3) $g_u = \cancel{q^{1/12}} \prod_{n=0}^{\infty} (1 - q^n \zeta_{p^r}^u)$

$\cdot \prod_{n=1}^{\infty} (1 - q^n \zeta_{p^r}^{-u})$

$\xmapsto{q \mapsto 0} 1 - \zeta_{p^r}^u$.

$g_u \cup g_v \xmapsto{\infty} (1 - \zeta_{p^r}^u, 1 - \zeta_{p^r}^v)_r$

$\Pi = \infty \circ Z$.

4) ∞ factors through quot. by I. //

Recall: $^{\sigma}J = \varprojlim H^1_{et}(X_1(p^r)_{/\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes$ ord

$$\begin{tikzcd}
0 \arrow[r] & {}^{\sigma}J_{sub} \arrow[r] \arrow[d, phantom, "\text{inertia}\ \wr\wr"] & {}^{\sigma}J \arrow[r] & {}^{\sigma}J_{quo} \arrow[r] \arrow[d, phantom, "\wr\wr"] & 0 \\
& h^2(1) & & \mathcal{C} & G_{\mathbb{Q}_p}
\end{tikzcd}$$

$\uparrow$ Λ-adic Hecke $\qquad$ $\uparrow$ Λ-adic cusp

$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad c : G_{\mathbb{Q}} \to \mathrm{Hom}_h({}^{\sigma}J_{sub}, {}^{\sigma}J_{quo})$

$\rightsquigarrow \Psi_c : X_\infty^-(1) \twoheadrightarrow \mathcal{C}/I\mathcal{C}$

$\mathrm{Hom}_h({}^{\sigma}J_{sub}, {}^{\sigma}J_{quo}) \cong {}^{\sigma}J_{quo}$

as h-mods.

${}^{\sigma}J_{sub} \oplus \mathcal{C}$

<u>Thm</u> (Ohta) $C = {}^{\sigma}\mathcal{J}_{quo}$.

Fact: when we reduce mod I things become "canonical":

- $S/IS \cong {}^{\sigma}\mathcal{J}^{+}/I\,{}^{\sigma}\mathcal{J}^{+} \xrightarrow{\sim} {}^{\sigma}\mathcal{J}_{quo}/I\,{}^{\sigma}\mathcal{J}_{quo}$

  $\subseteq G/IG$.

- ${}^{\sigma}\mathcal{J}_{sub}/I\,{}^{\sigma}\mathcal{J}_{sub} \cong (h/I)^{2}(1)$

  $\cong (\tilde{\Lambda}^{+}/\xi)^{2}(1)$ $\quad \xi = (\xi_k) \in \tilde{\Lambda}^{+}$

$\rightsquigarrow$ $\xi \,\& \,\psi_c$ induces a canonical map. $\Upsilon : X_{\infty}^{-}(1) \longrightarrow S/IS$ of $\tilde{\Lambda}$-mods.

<u>Conj</u> (S.) $\exists\, u \in \tilde{\Lambda}^{\times}$ s.t.

$$\Upsilon \circ \varpi = u, \quad \varpi \circ \Upsilon = u.$$

<u>Strong form</u>: $u = 1$, i.e.,

Υ & ϖ are inverse maps

<u>Def</u> $\xi'_k \in \Lambda$ s.t.

$$\xi'_k(\nu^s - 1) = L'_p(\omega^k, s-1)$$

$$(\xi_k(\nu^s - 1) = L_p(\omega^k, s-1)).$$

$$\xi' = (\xi'_k) \in \tilde{\Lambda}^{+} \qquad \hat{\Lambda} = \mathbb{Z}_p[[\mathbb{Z}_p^{\times}]]$$

Theorem (Fukaya-Kato)

$$\xi' \Upsilon \circ \varpi = \xi'.$$

Rmks: 1) Result was on $S/IS \otimes \mathbb{Q}_p$.
Improved by FKS. Also follows from Ohta that S/IS has no p-torsion.

2) $X_\infty^{(1-k)}$ pseudo-cyclic
$\Longrightarrow$ weak form of conj. in ω^{2-k}-eigenspace.
$\Longrightarrow$ conj. of McCallum-S.

3) If ξ_k has no irred. square factors, then strong form holds.

4) $\mathfrak{h}_{\mathfrak{Z}}$ Gorenstein

if $\mathfrak{M} = \mathrm{Hom}_{\Lambda}(\mathfrak{h}_{\mathfrak{Z}}, \Lambda)$ is

free over $\mathfrak{h}_{\mathfrak{Z}}$, so $\cong \mathfrak{h}_{\mathfrak{Z}}$.

Wake, Wang-Erickson:

if $X_{\infty}^{(2-k)} = 0 \Rightarrow \mathfrak{h}_{\mathfrak{Z}_k}$ Goren.

$\Longrightarrow S/IS \cong I/I^2$.

5) $\mathfrak{h}_k$ Gorenstein

$\Longleftrightarrow X_{\infty}^{(1-k)}$ cyclic

(Ohta, extends WWE).

Why ξ'?

On left: "$\log \chi_p \cup$"

On right: p-adic regulator of $g_u \cup g_v$.

<u>Alternate form:</u>

$\exists$ $\tilde{\Lambda}$-homom. $\Phi : \mathfrak{X}_\infty^-(-1) \to \tilde{\Lambda}^+$

$\ker \cong \alpha(X_\infty^+)$

$\operatorname{coker} \cong (X_\infty^+[p^\infty])^\vee$.

$$\Upsilon \otimes \Phi : X_\infty^- \cancel{(1)} \otimes_{\mathbb{Z}_p} \mathfrak{X}_\infty^- \cancel{(-1)} \longrightarrow S/IS \otimes_{\mathbb{Z}_p} \tilde{\Lambda}^+ .$$

$$\Psi_\infty : \mathcal{C}_\infty \longrightarrow X_\infty^- \otimes \mathfrak{X}_\infty^-$$

universal Mazur–Tate elt.

$$\mathcal{L} \in S^{ord} \hat{\otimes}_{\mathbb{Z}_p} \tilde{\Lambda}^+$$

$$\mathcal{L} = \varprojlim_r \sum_{a=1}^{p^r-1} U_p^{-r} \{\infty \to \tfrac{a}{p^r}\} [a]_r$$

$\rightsquigarrow$ Mazur–Kitagawa
2-var. p-adic L fn.

Equiv form of conj.:

$$1-\xi = (1-\xi_{p^r})_r$$

$$\Upsilon \otimes \Phi (\varpi_\infty (1-\xi)) = \overline{\mathcal{L}}$$

where $\overline{\mathcal{L}} = \mathcal{L} \bmod I$

Cor $G = \mathcal{X}_\infty - \mathrm{Gal}(M_\infty / F_\infty)$

Suppose $X_\infty^+ = 0$. Then

$\exists$ ex. seq.

$$0 \to \frac{I_G X'_{M_\infty}}{I_G^2 X'_{M_\infty}} \to \frac{X_\infty^- \otimes \mathcal{X}_\infty^-}{\langle \Psi_\infty(1-S) \rangle} \to (X_\infty^-)^{\otimes 2} \to 0.$$

$$\frac{S/I_S \otimes \tilde{\Lambda}^+}{\overline{\mathcal{L}}}$$