$X_\infty^-(1)$ ~ how to construct elts. ①

$F_r = \mathbb{Q}(\mu_{p^r})$, $S = \{\text{primes} \mid p\}$
$= \{(1-\zeta_{p^r})\}$

$\mathfrak{g}_r = G_{F_r,S}$ = Galois gp. of max'l S-ramified extn.

$\mathcal{O}_r = \mathcal{O}_{F_r,S} = \mathbb{Z}[\mu_{p^r}, \frac{1}{p}]$

$A_r = Cl_{F_r} \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

<u>Kummer theory</u>:

$\mathcal{B}_r = \{a \in F_r^\times \mid a\mathcal{O}_r = \mathfrak{a}^{p^r},$
$\mathfrak{a} \subset \mathcal{O}_r$ ideal$\}$

$H^1(\mathfrak{g}_r, \mu_{p^r}) = \mathcal{B}_r / F^{\times p^r}$

$$0 \to \mathcal{O}_r^\times / \mathcal{O}_r^{\times p^r} \to H^1(g_r, \mu_{p^r}) \quad (2)$$

$$\to A_r[p^r] \to 0.$$

$$H^2(g_r, \mu_{p^r}) \cong A_r / p^r A_r.$$

<u>Cup products</u>

$$H^1(g_r, \mu_{p^r}) \otimes H^1(g_r, \mu_{p^r})$$

$$\xrightarrow{\cup} H^2(g_r, \mu_{p^r}^{\otimes 2})$$

$$\cong H^2(g_r, \mu_{p^r}) \otimes_{\mathbb{Z}} \mu_{p^r}$$

$$\rightsquigarrow (\ ,\ )_r : \mathcal{O}_r^\times \times \mathcal{O}_r^\times \to A_r \otimes_{\mathbb{Z}} \mu_{p^r}$$

Theorem (McCallum–S.)

Let $a, b \in \mathcal{O}_r^\times$. Let $E_r = F_r(\sqrt[p^r]{a})$, $G = \mathrm{Gal}(E_r/F_r)$. Write $b = N_{E_r/F_r}\, y$, $y\,\mathcal{O}_{E_r,S} = \mathfrak{C}^{1-\sigma}$ where $\mathfrak{C} \subset \mathcal{O}_{E_r,S}$ ideal, $\sigma \in G$.

Then
$$(a,b)_r = [N_{E_r/F_r}\mathfrak{C}] \otimes \frac{\sigma(\sqrt[p^r]{a})}{\sqrt[p^r]{a}}.$$

Cor $(a, 1-a)_r = 0$ if $a, 1-a \in \mathcal{O}_r^\times$.

Proof: $1-a = N_{E_r/F_r}\underbrace{(1-\sqrt[p^r]{a})}_{\in\, \mathcal{O}_{E_r,S}^\times}$. //

Ex $\zeta = \zeta_{p^r}$ 1) $\zeta^a + (1-\zeta^a) = 1$

2) $$\frac{\zeta^a \pm 1}{\zeta^{a+b}-1} \pm \zeta^a \frac{\zeta^b \pm 1}{\zeta^{a+b}-1} = -1$$

3) in blue

Remark In all known cases w/ $p \mid \frac{B_k}{k}$, $A_1^{(1-k)} \otimes \mu_p \cong \mathbb{F}_p(2-k)$,

$\Delta = \mathrm{Gal}(F_1/\mathbb{Q})$

McCallum–S.: For $p < 25{,}000$, $k < p$ w/ $p \mid B_k$, $\exists$ a unique (up to nonzero scalar) nonzero pairing $\mathcal{O}_1^\times \times \mathcal{O}_1^\times \to \mathbb{F}_p(2-k)$

That is bilinear, antisymm., Galois-equivariant & "satisfies 3".

<u>Conj</u> C_r = cyclotomic p-units in $\mathcal{O}_r^\times$.

$C_r \otimes C_r \xrightarrow{(\ ,\ )_r} A_r^- \otimes \mu_{p^r}$

is surjective.

<u>Uses</u>:

1) Relations in max'l pro-p quotient of $\mathcal{G}_r$.

2) p-parts of class gps. over p-ramified Kummer extensions of F_r.

<u>First</u>: $\cdot\ H^1(\mathcal{G}_r, \mathbb{Z}_p(1)) \cong \mathcal{O}_r^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$

$\cdot\ H^2(\mathcal{G}_r, \mathbb{Z}_p(1)) \cong A_r$.

Note: $H^2(\mathcal{G}_r, \mathbb{Z}_p(2)) \underset{\text{in gen.}}{\not\cong} A_r \otimes \mathbb{Z}_p(1)$

Iwasawa cohomology:

T compact $\mathbb{Z}_p[[G_{\mathbb{Q},S}]]$-module.

$$H^i_{Iw}(F_\infty, \underset{T}{\cancel{\mathbb{Z}_p(1)}}) = \varprojlim_{cor} H^i(\underset{\mathcal{G}_r}{\underset{\shortparallel}{G_{F_n,S}}}, T)$$

These are $\tilde{\Lambda} = \mathbb{Z}_p[[\tilde{\Gamma}]]$,

$\tilde{\Gamma} = \mathrm{Gal}(F_\infty/\mathbb{Q})$ - modules.

Ex $H^1_{Iw}(F_\infty, \mathbb{Z}_p(1)) \cong \mathcal{E}_\infty$

$$= \varprojlim \mathcal{O}_v^\times \otimes \mathbb{Z}_p$$

$$H^2_{Iw}(F_\infty, \mathbb{Z}_p(1)) \cong X_\infty.$$

$$(2) \qquad X_\infty(1).$$

$$H^2_{Iw}(F_\infty, \mathbb{Z}_p(2))_{\Gamma^{p^{r-1}}} \cong \cancel{}.$$

$$H^2_{\cancel{Iw}}(\mathcal{G}_r, \mathbb{Z}_p(2))$$

Cup prods.

$$\begin{tikzcd}
H^1(\mathcal{G}_{r+1}, \mathbb{Z}_p(1)) \arrow[d, "\mathrm{cor}"] & \times \, H^1(\mathcal{G}_{r+1}, \mathbb{Z}_p(1)) \arrow[r] & H^2(\mathcal{G}_{r+1}, \mathbb{Z}_p(2)) \arrow[d, "\mathrm{Cor}"] \\
H^1(\mathcal{G}_r, \mathbb{Z}_p(1)) & \times \, H^1(\mathcal{G}_r, \mathbb{Z}_p(1)) \arrow[r] \arrow[u, "\mathrm{Res}"'] & H^2(\mathcal{G}_r, \mathbb{Z}_p(2))
\end{tikzcd}$$

$$(\mathrm{Cor}\, x, y) = \mathrm{Cor}(x, \mathrm{Res}\, y).$$

$$\rightsquigarrow (\ ,\ )_\infty : \widehat{\mathcal{O}_\infty^\times} \times \mathcal{E}_\infty \to X_\infty(1)$$

$$\mathcal{O}_\infty = \bigcup \mathcal{O}_r \qquad \widehat{\mathcal{O}_\infty^\times} = \varprojlim \mathcal{O}_\infty^\times / \mathcal{O}_\infty^{\times p^n}$$

$$X_\infty(1)^+ = X_\infty^-(1)$$

Given an abelian p-ramified extn. M_∞/F_∞ Galois $/\mathbb{Q}$, $G = \mathrm{Gal}(M_\infty/F_\infty)$.

(Ex $G = \mathfrak{X}_\infty$; $M_\infty = F(p^\infty\sqrt{p})$)

I_G augmentation ideal in $\mathbb{Z}_p[[G]]$

ex. seq.

$$0 \to G \to \frac{\mathbb{Z}_p[[G]]}{I_G^2} \to \mathbb{Z}_p \to 0$$

$$g \longmapsto g-1$$

$$g \longmapsto 1$$

$$\rightsquigarrow H^1_{Iw}(F_\infty, \mathbb{Z}_p(1)) \xrightarrow{\partial} H^2_{Iw}(F_\infty, G(1))$$

$$\cong H^2_{Iw}(F_\infty, \mathbb{Z}_p(1)) \hat{\otimes}_{\mathbb{Z}_p} G.$$

S-reciprocity map

$$\Phi_{M_\infty/F_\infty} : \mathcal{E}_\infty \to X_\infty \otimes_{\mathbb{Z}_p} G.$$

Lemma If $\chi_a : G \to \mathbb{Z}_p(1)$, $a \in \hat{\mathcal{O}}_\infty^\times$,

$$\left(\frac{\sigma(\sqrt[p^n]{a})}{\sqrt[p^n]{a}} = \chi(\sigma) \right)$$

then $(1 \otimes \chi_a) \circ \Phi_{M_\infty/F_\infty}(b)$

$= (a, b)_\infty$ for $b \in \mathcal{E}_\infty$.

What is the structure of

X_{M_∞} = unram. Iwasawa module / M_∞.

$\downarrow$

X'_{M_∞} = completely split Iwasawa module

= max'l quot. in which all primes $| p$ split completely.

$$0 \to X'_{M_\infty} / I_G X'_{M_\infty} \longrightarrow X_\infty \to G^{ur} \to 0$$

G^{ur} = Galois gp. of max'l unram. subextn. of M_∞ / F_∞.

<u>Thm</u> (S.) $\exists$ exact seq. of $\tilde{\Lambda}$-mods.

$$0 \to \frac{I_G X'_{M_\infty}}{I_G^2 X'_{M_\infty}} \to \frac{X_\infty \otimes G}{\Phi_{M_\infty/F_\infty}(\mathcal{E}_\infty)} \to (G^{un})^{\otimes 2} \to 0.$$

<u>Ex</u> $M_\infty = F_\infty(\sqrt[p^\infty]{p})$

$p = 37$. $A_F = A_F^{(5)} \cong \mathbb{F}_p$

$\rightsquigarrow X_\infty^{(5)} \cong \mathbb{Z}_p$.

$\chi_p : G \xrightarrow{\sim} \mathbb{Z}_p(1)$

tot. ramified at p

$$X'_{M_\infty}/I_G X'_{M_\infty} \cong X_\infty.$$

$$\frac{I_G X'_{M_\infty}}{I_G^2 X'_{M_\infty}} \cong \frac{X_\infty(1)}{\langle (p, 1-\zeta)_\infty \rangle}$$

$$1-\zeta = (1-\zeta_{p^r})_r$$

$$(p, 1-\zeta_p)_1 \neq 0$$

$$\Rightarrow I_G X'_{M_\infty} = 0,$$

$$\Rightarrow I_G X_{M_\infty} = 0,$$

$$\Rightarrow X_{M_\infty} \cong (X_{M_\infty})_G \cong X_\infty$$

$$X_{M_\infty} \cong \mathbb{Z}_p$$

$A_{\mathbb{Q}(\mu_{37}, \sqrt[37]{37})} \cong \mathbb{Z}/37\mathbb{Z}$ 13