p odd prime ①

$F_\infty = \mathbb{Q}(\mu_{p^\infty})$ cyclotomic

$\mathbb{Z}_p$-extn. of $F = \mathbb{Q}(\mu_p)$

$\Gamma = \mathrm{Gal}(F_\infty/F) \cong 1 + p\mathbb{Z}_p = \overline{\langle u \rangle}$.

$\Delta = \mathrm{Gal}(F_\infty/\mathbb{Q}_\infty) \cong (\mathbb{Z}/p\mathbb{Z})^\times \underset{\omega}{\hookrightarrow} \mathbb{Z}_p^\times$

X_∞ unramified Iwasawa module $/ F_\infty$

f.g. torsion Λ-module:

$$\Lambda = \mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$$

$$\gamma - 1 \longmapsto T \quad \gamma = [u].$$

Iwasawa main conjecture ②

k even $\not\equiv 0 \mod p-1$

$\mathrm{char}_\Lambda X_\infty^{(1-k)} = (f_k)$,

$f_k(u^s - 1) = L_p(\omega^k, s) \ \forall s \in \mathbb{Z}_p$

Hida theory $\quad \Lambda = \mathbb{Z}_p[[T]]$

Def A Λ-adic modular form of char ω^{k-2} is a power series $\mathcal{F} \in Q(\Lambda) + \Lambda[[q]]$ s.t. $\forall\, \varepsilon : \Gamma \to \overline{\mathbb{Q}}_p^\times$ fin. order, $j \geq 2$, $\mathcal{F}(\varepsilon(u) u^{j-2} - 1)$ is a modular form

$Q(\Lambda) =$ quot. field

of level p^r for $\ker \mathcal{E} = \Gamma_{p^{r-1}}$, (3)
weight j, character $\mathcal{E}\omega^{k-j}$.

<u>Ex</u> Λ-adic Eisenstein series

$$\mathcal{E}_k = \frac{c_k}{2} + \sum_{n=1}^{\infty} \Big(\sum_{\substack{d \mid n \\ p \nmid d}} d\omega^{k-2}(d)$$

$$\cdot (1+T)^{\log_v d} \Big) q^n, \quad d\omega^{-1}(d) = v^{\log_v (d)}$$

where $c_k(v^s - 1) = L_p(\omega^k, -1-s)$
$\forall s \in \mathbb{Z}_p$.

$$\mathcal{E}_k(v^j - 1) = E_{j,\omega^{k-j}}$$

level p, wt. j. char ω^{k-j}.

$\mathcal{F}$ is a cusp form if all of its specializations are.

U_p-operator on

$\mathcal{M} = \{\Lambda\text{-adic forms}\}$

$\cup$

$\mathcal{G} = \{\Lambda\text{-adic cusp forms}\}$

$a_n(U_p \mathcal{F}) = a_{np}(\mathcal{F})$.

$\boxed{\begin{array}{l} \mathcal{F}\ U_p \\ \text{-eigenvec.} \\ \text{ord. if} \\ a_p(\mathcal{F}) \in \Lambda^\times \end{array}}$

Hida's idempotent:

$$e = \lim_{n \to \infty} U_p^{n!}$$

$\mathcal{M}^{ord} = e\mathcal{M}$, $\mathcal{G}^{ord} = e\mathcal{G}$

<u>Thm</u> (Hida)

$\mathbb{M}^{ord}, \mathbb{G}^{ord}$ f.g. free Λ-modules.

$$\begin{tikzcd}
\mathfrak{h}^{ord} \subset \mathrm{End}_\Lambda(\mathbb{M}^{ord}) \arrow[d] \\
h^{ord} \subset \mathrm{End}_\Lambda(\mathbb{G}^{ord})
\end{tikzcd}$$

Λ-adic Hecke algebras

Hida duality: Perfect pairings

$$\begin{tikzcd}
\mathfrak{h}^{ord} \times \mathbb{M}^{ord} \arrow[r] \arrow[d] & \Lambda \arrow[d, equal] & (T, \mathcal{F}) \\
h^{ord} \times \mathbb{G}^{ord} \arrow[r] \arrow[u, hook, shift left=8] & \Lambda & \mapsto a_1(T\mathcal{F})
\end{tikzcd}$$

Control Theorems:

$$\mathfrak{h}^{ord}/\omega_{k,r}\,\mathfrak{h}^{ord} \xrightarrow{\sim} h_k(p^r,\mathbb{Z}_p)^{ord}$$

$$\mathfrak{m}^{ord}/\omega_{k,r}\,\mathfrak{m}^{ord} \xrightarrow{\sim} M_k(p^r,\mathbb{Z}_p)^{ord}$$

$$\omega_{k,r} = (1+T)^{p^{r-1}} - u^{p^{r-1}(k-2)} \in \Lambda$$

Get ordinary parts of modular forms and Hecke algebras of all weights & levels!.

Also for cusp forms, cuspidal Hecke algs.

$$H^1_{\text{ét}}(X_1(p^r)_{/\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \circlearrowleft G_{\mathbb{Q}}$$

$\circlearrowleft$

$h_2(p^r, \mathbb{Z}_p)$ wt. 2 level p^r

(dual action)

$$\mathcal{J}^{ord} = \varprojlim H^1_{\text{ét}}(X_1(p^r), \mathbb{Z}_p(1))^{ord}$$

Thm (Hida, Ohta)

$\mathcal{J}^{ord}$ has h^{ord}-rank 2, free/Λ. $\exists$ exact seq. of $h^{ord}[G_{\mathbb{Q}_p}]$-modules

$$0 \to \mathcal{J}^{ord}_{sub} \to \mathcal{J}^{ord} \to \mathcal{J}^{ord}_{quo} \to 0$$

s.t. on inertia at p,

$$\mathcal{J}^{\mathrm{ord}}_{\mathrm{sub}} \cong \left(h^{\mathrm{ord}}\right)^{\iota}(1)$$

Galois elt. $\leftrightarrow j$ acts as $\langle j \rangle$.

$$\mathcal{J}^{\mathrm{ord}}_{\mathrm{quo}} \cong \mathcal{G}^{\mathrm{ord}}$$ w/ trivial action

Remarks:

1) $\rho : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Q}(h^{\mathrm{ord}}))$

$$\rho|_{\mathrm{inertia}} = \begin{pmatrix} \det \rho & * \\ 0 & 1 \end{pmatrix}$$

$\det \rho(\varphi_\ell) = \ell \langle \ell \rangle, \quad \ell \neq p.$

$\mathrm{tr}\, \rho(\varphi_\ell) = T_\ell, \quad \ell \neq p.$

2) $\sigma J^{ord} / \omega_{z,r}\, \sigma J^{ord}$

$\cong H^1(X_1(p^r), \mathbb{Z}_p(1))^{ord}$

f ord. newform of level p^r, wt. 2

$\phi_f : h_2(p^r, \mathbb{Z}_p) \longrightarrow \overline{\mathbb{Q}_p}$

$I_f = \ker \phi_f$.

$$\frac{H^1(X_1(p^r), \mathbb{Z}_p)^{ord}}{I_f\, H^1(X_1(p^r), \mathbb{Z}_p)^{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = V_f$$

V_f 2-dim'l $/ K_f = \mathbb{Q}_p(\{a_n(f)\})$.

Eisenstein ideal

$I_k \subset \mathfrak{h}^{ord}$ generated by

$T_\ell - 1 - \ell\langle\ell\rangle$ $\ell \neq p$,

$U_p - 1$, $\langle j\rangle - \omega^{k-2}(j)$,

$j \in (\mathbb{Z}/p\mathbb{Z})^\times$.

Consider $\mathfrak{h}^{ord}$ as a Λ-mod.

by ~~&~~ γ acts as $\langle u\rangle^{-1}$.

$$\mathfrak{h}^{ord}/I_k \cong \Lambda/(\xi_k)$$ Mazur-Wiles

$$\xi_k(u^s-1) = L_p(\omega^k, s-1).$$

(11)

$\mathfrak{m}_k$ max'l ideal of $\mathfrak{h}^{\mathrm{ord}}$ containing I_k:

$\mathfrak{m}_k = I_k + (p, \langle u \rangle - 1)$.

$h = h_k := h^{\mathrm{ord}}_{\mathfrak{m}_k}$.

$\bar{J} = \bar{J}_k := \bar{J}^{\mathrm{ord}}_{\mathfrak{m}_k}$.

$\bar{J} \otimes_\Lambda Q(\Lambda)$ free of rank 2 over $h \otimes_\Lambda Q(\Lambda)$.

$\rho : G_{\mathbb{Q}} \to \mathrm{Aut}_h(\bar{J})$

view this as acting on $\bar{J}_{sub} \oplus \bar{J}_{quo}$

$$\rho|_{\text{inertia}} = \begin{pmatrix} \det\rho & b \\ 0 & 1 \end{pmatrix}$$

$$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

${}^{\sigma}J / I\,{}^{\sigma}J$ may look like

$$\begin{pmatrix} \det\rho & \overset{\neq 0}{*} \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \det\rho & 0 \\ \underset{0 \neq}{*} & 1 \end{pmatrix}.$$

$B = \mathrm{Span}(\mathrm{im}\ b)$

$C = \mathrm{span}(\mathrm{im}\ C)$.

Facts
$\sigma, \tau \in G_{\mathbb{Q}}$

$a(\sigma) - \det\rho(\sigma) \in I_k$

$d(\sigma) - 1 \in I_k$

$b(\sigma)c(\tau) \in I_k$

Consider $(I = I_k)$

$\bar{c} = c \bmod I : G_{\mathbb{Q}} \to C/IC$

<u>Claim</u>: $\psi_c = (\det \rho)^{-1} \cdot \bar{c}$ is a 1-cocycle.

<u>Proof</u>. $c(\sigma\tau) = c(\sigma)a(\tau) + d(\sigma)c(\tau)$

$\equiv c(\sigma)\det\rho(\tau) + c(\tau)$ mod IC.

$\rightsquigarrow \psi_c$ cocycle.

Homom. on $G_{\mathbb{Q}(\mu_{p^\infty})}$.

$\psi_c : G_{\mathbb{Q}(\mu_{p^\infty})} \twoheadrightarrow C/IC$

homom., unramified everywhere.

$\rightsquigarrow$ factors through

$X_\infty \longrightarrow C/IC.$

Actually, factors through

$X_\infty^{(1-k)}(1) \twoheadrightarrow C/IC.$

homom. of Λ-modules

C faithful h-mod.

$\Longrightarrow$ $\mathrm{char}_\Lambda(C/IC)$ divis. by

$\mathrm{char}_\Lambda(h/I) = (\xi)$

(22)

$\Rightarrow \mathrm{char}_\Lambda(X_\infty^{(1-k)}(1))$

divis. by (ξ_k)

$\Rightarrow \mathrm{char}_\Lambda(X_\infty^{(1-k)})\,\Phi(f_k)$.

divis. by

Divis. for all k even

$\rightsquigarrow$ equality $\rightsquigarrow$

IMC //