Last time: BF elt

for $m = 1$

$\iota : Y_1(N) \hookrightarrow Y_1(N)^2$

${}_c g_{0,\frac{1}{N}} \in \mathcal{O}(Y_1(N))^\times$

$$BF_{1,N} := \iota_*\left(\kappa_p\left({}_c g_{0,\frac{1}{N}}\right)\right)$$

$$\in H^3_{\text{ét}}(Y_1(N)^2, \mathbb{Z}_p(2)).$$

Today: classes $/\mathbb{Q}(\mu_m)$.

Kato's modular curves

Idea: "find $\operatorname{Spec}\mathbb{Q}(\mu_m)$ in a modular curve."

$U \subset GL_2(\mathbb{A}_f)$

$\rightsquigarrow Y(U)$ mod curve $/\mathbb{Q}$

$$Y(U)(\mathbb{C}) = GL_2^+(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times \mathcal{H} / U$$

may have many components.

Components def / cyclo fields.

<u>Def^n</u> $M, N \in \mathbb{Z}_{\geq 1}$

$$U(M,N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathbb{Z}}) : \begin{array}{l} a \equiv 1,\ b \equiv 0 \bmod M \\ c \equiv 0,\ d \equiv 1 \bmod N \end{array} \right\}$$

$$= 1 \bmod \begin{pmatrix} M & M \\ N & N \end{pmatrix}.$$

$A \geq 1$

$$U(M(A), N) := 1 \bmod \begin{pmatrix} M & MA \\ N & N \end{pmatrix}.$$

Cf Kato, Asteisque 295.

$Y(M,N)$, $Y(M(A),N)$.

Always take $M \mid N$
(resp $MA \mid N$).

<u>Fact</u> $\exists$ map of $\mathbb{Q}$-varieties

$$S_M : Y(M, MN) \twoheadrightarrow Y_1(N) \times \operatorname{Spec} \mathbb{Q}(\mu_m).$$

Given by action of $\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$
($\tau \mapsto \tau/M$ on $\mathcal{H}$).

Strategy: Build elts on

$$Y(M,N) \underset{\operatorname{Spec}\mathbb{Q}(\mu_m)}{\times} Y(M,N).$$

Lemma $(M \mid N)$

(a) $\mathcal{U}(M,N)$ normalized by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(b) If ℓ prime, largest subgp of $\mathcal{U}(M(\ell),N)$ norm$^{\text{zed}}$ by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is $\mathcal{U}(M\ell,N)$.

Def^n

- Let $\iota_{M,N}$ be composite map

$$Y(M,N) \overset{\text{diag}}{\hookrightarrow} Y(M,N)^2 \xrightarrow{\left(1,\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)} Y(M,N)^2.$$

- Let ${}_c\mathrm{REis}_{M,N} = (\iota_{M,N})_* \left(K_p\left({}_c g_{0,\frac{1}{N}}\right)\right)$

$\cdot\ {}_c BF_{M,N} = (s_M \times s_M)_* ({}_c REis_{M,MN}).$

$$\in H^3_{ét}(Y_1(N)^2_{\mathbb{Q}(\mu_M)}, \mathbb{Z}_p(2)).$$

<u>Norm rel^{n}s</u> Cf. §4.2 of notes

<u>Easy norm rel^n</u>: pushfwd of ${}_c REis_{M,N\ell}$ via quotient map

$$Y(M,N\ell)^2 \to Y(M,N)^2$$

is ${}_c REis_{M,N}$ if $\ell \mid N$.

(Easy consequence of norm-compat of Siegel units)

Doesn't work for quot map

$$Y(M\ell, N)^2 \longrightarrow Y(M,N)^2 :$$

pushfwd of ${}_c\mathrm{REis}_{M\ell,N}$ is

$$\ell^2 \cdot {}_c\mathrm{REis}_{M,N}. \qquad \text{(bad!)}$$

$$T_\ell : \quad Y(M\ell, N) \rightarrow Y(M,N)$$

given by action of $\begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix}$.

<u>Thm</u> if $\ell \mid M$ and $M \mid N$,

$$(T_\ell \times T_\ell)_* \left({}_c\mathrm{REis}_{M\ell,N} \right)$$

$$= (U'_\ell, U'_\ell) \cdot {}_c\mathrm{REis}_{M,N}.$$

Corollary for $\ell \mid M$ and $\ell \mid N$,

$$\mathrm{cores}^{\mathbb{Q}(\mu_{M\ell})}_{\mathbb{Q}(\mu_M)}\left({}_c BF_{M\ell, N}\right)$$

$$= (U_\ell', U_\ell') \cdot {}_c BF_{M,N}.$$

$$
\begin{tikzcd}
Y(M\ell, N) \arrow[r, hook, "\iota_{M\ell,N}"] \arrow[d, equal] & Y(M\ell, N)^2 \arrow[d] \arrow[rd, bend left, "\tau_\ell \times \tau_\ell"] & \\
Y(M\ell, N) \arrow[r, hook, "\iota_{M(\ell),N}"] \arrow[d, "(\deg \ell^2)"'] \arrow[dr, phantom, "\Diamond"] & Y(M(\ell), N)^2 \arrow[d, "(\deg \ell^2)"'] \arrow[r, "\begin{pmatrix}1&0\\0&\ell\end{pmatrix}^2"] & Y(M,N)^2 \\
Y(M,N) \arrow[r, hook, "\iota_{M,N}"] & Y(M,N)^2 &
\end{tikzcd}
$$

<u>Key Facts</u>

i) $\iota_{M(\ell),N}$ is a closed embedding.

(Part b of Lemma).

ii) $\Diamond$ is <u>Cartesian</u>

(both vertical maps degree ℓ^2).

Push-pull lemma:

for any Cartesian diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"'] & Y \arrow[d, "\gamma"] \\ Z \arrow[r, "\delta"'] \arrow[ur, dotted] & W \end{tikzcd}$$

$$\alpha_* \cdot \beta^* = \gamma^* \cdot \delta_* .$$

Applied to $(\Diamond)$, + coh. class

$K_p(g_{0,\frac{1}{N}}) \in H^1_{\text{ét}}(Y(M,N), \mathbb{Z}_p(1))$

Conclude that 2 classes in

$H^3_{\text{ét}}(Y(M(\ell),N)^2, \mathbb{Z}_p(2))$ agree:

- pullback of ${}_c\mathrm{REis}_{M,N}$
- pushfwd of ${}_c\mathrm{REis}_{M_\ell,N}$.

Push forward along diag^l arrow:

$$(\iota_\ell \times \iota_\ell)_*\left({}_c\mathrm{REis}_{M_\ell,N}\right) =$$

$$(u_\ell', u_\ell')\cdot {}_c\mathrm{REis}_{M,N}. \qquad \square.$$

Projection to eigenforms

f, g eigenforms level N
(not necessarily new) $\boxed{\text{wt } 2}$

$p \mid N$.

Assume $p \nmid a_p(f), a_p(g)$.

Can construct $V_p(f)^* \otimes V_p(g)^*$
as

$$H^2_{\text{ét}}\left(Y_1(N)^2_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(2)\right) \otimes E \Big/ \left\{ (T'_\ell, 1) - a_\ell(f), \ (1, T'_\ell) - a_\ell(g), \ \forall \ell \text{ prime} \right\}$$

Thus $H^1(\mathbb{Q}(\mu_m), V_p(f)^* \otimes V_p(g)^*)$

is a quotient of

$$H^3_{ét}(Y_1(N)^2 \times \mathbb{Q}(\mu_m), \mathbb{Q}_p(2))$$

action of (U_p', U_p') is mult^n by $a_p(f)\,a_p(g)$.

Hence we can define

$$c_{p^r} = [a_p(f)a_p(g)]^{-r} \cdot \left(\text{image of } BF_{p^r, N}\right)$$

norm-compatible for $r \geqslant 1$.

+ denominators bounded

as $r \to \infty$.

($\longleftrightarrow$ class in $H^1(\mathbb{Q},$

$V_p(f)^* \otimes V_p(g)^* \otimes \Lambda)$)

Norm rel^n's for m not a power of p: involves degree 4 Euler factor.