A tool kit for building Euler systems

I Galois rep's in étale cohom.

K number field, X/K variety

$H^i_{et}(X_{\bar{K}}, \mathbb{Q}_p)$ has cts action of G_K

$\forall i \geq 0$

Fact: many interesting rep^{n}s appear as quotients of such H^i_{et}

Example 1: p-adic rep^{n} attached to a modular form

$f = \sum a_n q^n$ cuspidal modular eigenform of wt 2, level $\Gamma_1(N)$

$a_1 = 1$, $L = \mathbb{Q}(a_n : n \geq 1) \hookrightarrow \overline{\mathbb{Q}}$

$E/\mathbb{Q}_p$ finite containing L

$\Gamma_1(N) \rightsquigarrow$ modular curve $Y_1(N)/_{\mathbb{Q}}$

$\left(Y_1(N)(\mathbb{C}) = \Gamma_1(N) \backslash \mathcal{H} \right)$

$\rightsquigarrow H^1_{et}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)$ has actions of $G_{\mathbb{Q}}$ and Hecke algebra

Def^n: $V_p(f)$ = largest subspace of $H^1_{et}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_p) \otimes E$ st. $\forall \ell$, T_ℓ acts by a_ℓ

Prop: (i) $V_p(f)$ is 2-dim^l/E, irreducible

(ii) $V_p(f)$ is a direct summand of $H^1_{et} \otimes E$ as $G_{\mathbb{Q}}$-rep^n

(iii) $\forall \ell \nmid Np$, $V_p(f)$ unramified at ℓ and

$\det(1 - X\,\mathrm{Frob}_\ell^{-1} \mid V_p(f)) = 1 - a_\ell X + \ell\chi(\ell)X^2$

(iv) $V_p(f)^* \simeq V_p(f \otimes \chi^{-1})(1)$

Example 11: Rankin-Selberg convolutions

f, g cuspidal mod. eigenforms of wt 2, level $\Gamma_1(N)$

$E/\mathbb{Q}_p$ suff. large

$V = V_p(f) \otimes V_p(g)$ $G_{\mathbb{Q}}$ - direct summand of $H^2_{et}(Y_1(N)^2_{\bar{\mathbb{Q}}}, \mathbb{Q}_p) \otimes E$

Fact: $\forall \ell \nmid Np$, V unram at ℓ, and

$\det(1 - X\,\mathrm{Frob}_\ell^{-1} | V) =$ RS Euler factor at ℓ

II The Hochschild-Serre spectral sequence

Suppose we want to construct ES for V, $V =$ quot. of $H^i_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)$

$\rightsquigarrow$ want: a collection of elts in $H^1(\mathbb{Q}(\mu_m), H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)(n))$ for <u>some</u> $n \in \mathbb{Z}$ satisfying the ES norm rel^{n}s

(recall: ES for $V \Leftrightarrow$ ES for $V(n)$, any $n \in \mathbb{Z}$)

Thm (Jannsen): $K/\mathbb{Q}$ any number field, $\exists$ spectral seq.

$$E_2^{jk} = H^j(K, H^k_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)(n)) \Rightarrow H^{j+k}_{et}(X_K, \mathbb{Q}_p(n))$$

$\rightsquigarrow$ map

$$F^1 H^{i+1}_{et}(X_K, \mathbb{Q}_p(n)) \to H^1(K, H^i_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p(n)))$$

where

$$F^1 = \ker(H^{i+1}_{et}(X_K, \mathbb{Q}_p(n)) \to H^{i+1}_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(n)^{G_K})$$

$\Rightarrow$ to construct the ES, can try to construct collection of elt^s in $F^1 H^{i+1}_{et}(X_{\mathbb{Q}(\mu_m)}, \mathbb{Q}_p(n))$ some n

Example II (ctd): $V = V_p(f) \otimes V_p(g)$

$n \in \mathbb{Z}$

by above, want elements in

$$H^1(\mathbb{Q}(\mu_m), H^2_{et}(Y_1(N)^2_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(n))$$

$\rightsquigarrow$ want elements in

$F^1 H^3_{et}(Y_1(N)^2_{\mathbb{Q}(\mu_m)}, \mathbb{Q}_p(n))$

$= H^3_{et}$ (because $Y_1(N)^2$ affine surface, so $H^3_{et}(Y_1(N)^2_{\bar{\mathbb{Q}}}, \mathbb{Q}_p(n)) = 0$)

How do we construct elements in the H^3_{et}?

III Constructing elements in $H^k_{et}(X, \mathbb{Q}_p(n))$ $X/\mathbb{Q}$, $n \in \mathbb{Z}$

Idea: use tools from geometry

1) cup products:

$$H^i_{et}(X, \mathbb{Q}_p(m)) \times H^j_{et}(X, \mathbb{Q}_p(n))$$
$$\xrightarrow{\cup} H^{i+j}_{et}(X, \mathbb{Q}_p(m+n))$$

2) Kummer maps: if $f \in \mathcal{O}(X)^\times$, then $\kappa_p(f) \in H^1_{et}(X, \mathbb{Q}_p(1))$

3) pushforward: $Z \overset{\iota}{\hookrightarrow} X$ closed subvar, codim d

$$\rightsquigarrow \iota_* : H^i_{et}(Z, \mathbb{Q}_p(n))$$
$$\longrightarrow H^{i+2d}_{et}(X, \mathbb{Q}_p(n+d))$$

Example 11 (ctd):

two plausible approaches:

$n=3$: class in $H^3_{et}(Y_1(N)^2, \mathbb{Q}_p(3))$ from $\kappa_p(f_1) \cup \kappa_p(f_2) \cup \kappa_p(f_3)$,

$f_i \in \mathcal{O}(Y^2)^\times$

$n=2$: $Z = Y \xhookrightarrow{\text{diag}} Y^2$, $f \in \mathcal{O}(Y)^\times$

$\rightsquigarrow \iota_* \kappa_p(f) \in H^3_{et}(Y_1(N)^2, \mathbb{Q}_p(2))$

why should such a class belong to ES?

$$\begin{tikzcd}
H^3_{\mathcal{M}}(Y^2, \mathbb{Z}(2)) \arrow[d, "r_{et}"] \arrow[dr, "r_{\mathbb{C}}"] & \\
H^3_{et}(Y^2, \mathbb{Q}_p(2)) & (\mathrm{Fil}^1 H^2_{dR}(Y^2_{\mathbb{C}}))^\vee
\end{tikzcd}$$

$$H^2_{dR}(Y^2_{\mathbb{C}}))^\vee$$

$\exists$ commutative diagram

$$
\begin{tikzcd}
H^1_{\mathcal{M}}(Y, \mathbb{Z}(1)) \arrow[r, "\iota_{\mathcal{M},*}"] \arrow[d, "\kappa_p"] & {} \\
H^1_{et}(Y, \mathbb{Q}_p(1)) \arrow[r, "\iota_*"] & {}
\end{tikzcd}
$$

Fact: $\exists$ second regulator map (Beilinson reg.)

$r_{\mathbb{C}} : H^3_{\mathcal{M}}(Y^2, \mathbb{Z}(n)) \longrightarrow \big(\mathrm{Fil}^{[illegible]\,3-n}$

want for ES class:

$\langle \pi_{\mathbb{C}}(\text{this class}), \omega_{f,g} \rangle \sim$ leading term of $L(f \otimes g, s)\Big|_{s=3-n}$ at

certain diff^{l} attached to f, g

for $n=3$: noone knows how to do this

for $n=2$: works ($\leadsto$ ES of Beilinson-Flach elements)

what to push forward?

V Siegel units

Def^n: M cong. subgp $\leadsto Y(M)$

modular unit of level M = elt of $\mathcal{O}(Y(M))^\times$

Def^n: $\alpha, \beta \in \mathbb{Q}/\mathbb{Z}$, $(\alpha, \beta) = \left(\frac{a}{N}, \frac{b}{N}\right)$

$N \geq 1$, $0 \leq a < N$

$$g_{\alpha,\beta} : \mathcal{H} \longrightarrow \mathbb{C}$$

$$g_{\alpha,\beta}(\tau) = q^{w} \prod_{n \geq 0} \left(1 - q^{n + \frac{a}{N}} \zeta_N^{b}\right) \times$$

$$\prod_{n \geq 1} \left(1 - q^{n - \frac{a}{N}} \zeta_N^{-b}\right)$$

$q = e^{2\pi i \tau}$, $w = \frac{1}{12} - \frac{a}{N} + \frac{a^2}{2N^2}$

Let $c > 1$, $(c, 6N) = 1$

$$_c g_{\alpha,\beta} = \frac{g_{\alpha,\beta}^{c^2}}{g_{c\alpha, c\beta}}$$

IV constructing cohom. classes which are <u>interesting</u>

note: constructions for $n=2,3$ factor through motivic cohom.

e.g. $n=2$

$H^1_M(Y, \mathbb{Z}(1)) = \mathcal{O}(Y)^\times$

$H^3_M(Y^2, \mathbb{Z}(2)) =$

$\{\sum (Z_i, u_i), Z_i \subset Y^2 \text{ irred. curve}, u_i \in \mathbb{Q}(Z_i)^\times, \sum \operatorname{div}(u_i) = 0\} / \sim$

$\iota : Y \hookrightarrow Y^2$

$\rightsquigarrow \iota_{M,*} : g \longmapsto (Y, g)$

Prop^n: ${}_c g_{0,\frac{1}{N}}$ is a modular unit of level $\Gamma_1(N)$

Example 11 (ctd.):

$\iota : Y_1(N) \hookrightarrow Y_1(N)^2$

$\rightsquigarrow \mathcal{O}(Y_1(N))^\times \xrightarrow{\kappa_p} H^1_{et}(Y_1(N), \mathbb{Q}_p(1))$

$\quad \Psi$

$\quad {}_c g_{0,\frac{1}{N}} \longmapsto$

Thm (Bloch, Rankin):

$\langle r_{\mathcal{D}}(\iota_{H,*}({}_c g_{0,\frac{1}{N}})), \omega_{f,g} \rangle$

$\sim L'(f \otimes g, 1)$

$$\xrightarrow{\;L_*\;} H^3_{et}\left(Y_1(N)^2, \mathbb{Q}_p(2)\right)$$

$$\rotatebox[origin=c]{90}{$\in$}$$

$$\longrightarrow \; {}_c\mathcal{BF}_{1,N}$$

$$\|$$

$$\mathfrak{R}_{et}\left(L_{\mathcal{N},*}\left({}_c g_{0,\frac{1}{N}}\right)\right)$$