EULER SYSTEMS

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1) Galois reps + E.S.

2) Toolkit for bulding ES

3 + 4) 2 examples.

1) Galois rep^{n}s

K number fld

p prime

$E/\mathbb{Q}_p$ finite

Study rep^{n}s

$$\begin{array}{l} \rho : G_K \longrightarrow GL(V) \cong GL_d(E) \\ \quad \ \ \| \qquad\qquad\qquad\qquad\qquad\quad d < \infty. \\ \mathrm{Gal}(\bar{K}/K). \end{array}$$

Hypotheses:

1) ρ continuous

2) ρ "unram almost everywhere"

i.e. $\rho(I_v) = 1$ for all but fin. many primes v of K.

Examples. $(E = \mathbb{Q}_p)$

- Cyclotomic char

  $\chi : G_K \longrightarrow GL_1(\mathbb{Q}_p)$

  unram away from p.

- Tate modules

A/K ell curve

$$T_p(A) = \varprojlim_n A(\bar{K})[p^n]$$

$$V_p A = T_p A \otimes \mathbb{Q}_p \quad \text{2-dim}^{l}.$$

unram away from $p \cdot \mathrm{disc}(A)$.

- Étale coh. X/K variety

$X_{\bar{K}}$ base ext^n

$$H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p) \circlearrowleft G_K.$$

Def^n Say V **comes from geometry** if $\exists X/K$ and i,j st

V subquot of $H^i_{et}(X_{\bar{K}}, \mathbb{Q}_p)(j)$

$[V(j) = V \otimes \mathrm{cyclo}^j]$.

Tate modules:

$$V_p A = H^1_{et}(A_{\bar{K}}, \mathbb{Q}_p)(1)$$

1.2 L-functions

V Gal. rep^n $\quad$ v unramified prime

Def $P_v(V,T) =$

$$\det_E\left(1 - T\,\mathrm{Frob}_v^{-1} : V\right)$$

If V comes from geom can also define P_v for bad v

+ define

$$L(V,s) = \prod_v P_v(V, Nm(v)^{-s})^{-1}.$$

Eg if V = triv. rep

$L(V,s) = \zeta_K(s)$

(Dedekind ζ-fcn).

$V = H^1_{\acute{e}t}(A_{\bar{K}}, \mathbb{Q}_p)$

$\rightsquigarrow L(V,s)$ = Hasse-Weil $L(A,s)$.

~~Galois~~

Conj. $L(V,s)$ has meromorphic contn + funcl eqn.

1.3 Galois cohomology

Have E-vector spaces

$H^i(K, V)$ zero if $i > 2$.

$$H^0(K, V) = V^{G_K}$$

$$H^1(K, V) = \frac{\{\text{cts } \sigma : G_K \to V \text{ st } \sigma(gh) = \sigma(g) + g\sigma(h)\}}{\{\sigma \text{ of form } \sigma(g) = gv - v,\ v \in V\}}.$$

Not finite-dim^l in general.

<u>Kummer map</u>: $V = \mathbb{Q}_p(1)$

$$K^\times \otimes \mathbb{Q}_p \xhookrightarrow{\kappa} H^1(K, V)$$

$\exists$ also Kummer map for ell. curves.

Want to "cut these down to size".

<u>Def^n</u> <u>Local cond^n</u> for V at v

= any subspace $F_v \subset H^1(K_v, V)$
(finite-dim^l).

Eg

Strict cond^n $\quad \mathcal{F}_v^{str} = \{0\}$

relaxed $\quad \mathcal{F}_v^{rel} =$ everything

unramified $\quad \mathcal{F}_v^{ur} = \ker\left(H^1(K_v, V) \to H^1(I_v, V)\right)$

(best cond^n for $v \nmid p$)

Bloch-Kato (for $v \mid p$, V from geom)

$$\mathcal{F}_v^{BK} = \ker\left(H^1(K_v, V) \longrightarrow H^1(K_v, V \otimes \mathbb{B}_{cris})\right)$$

Def^n 3 Selmer groups:

- $\mathrm{Sel}_{\mathrm{rel}}(K, V) = \{ x \in H^1(K, V) :$

$$\mathrm{loc}_v(x) \in \mathcal{F}_v^{\mathrm{ur}} \quad v \nmid p,$$

$$\in \mathcal{F}_v^{\mathrm{rel}} \quad v \mid p \}.$$

- $\mathrm{Sel}_{\mathrm{BK}}(K, V)$, $\mathrm{Sel}_{\mathrm{str}}(K, V)$

  with BK, resp. strict, cond^s @ $v \mid p$.

  (Thm: these are finite-dim^l)

E.g. for $V = \mathbb{Q}_p(1)$:

$\mathrm{Sel}_{\mathrm{rel}}$ = image of

$$\mathcal{O}_K[\tfrac{1}{p}]^\times \otimes \mathbb{Q}_p$$

$\mathrm{Sel}_{\mathrm{BK}}$: $\mathcal{O}_K^\times \otimes \mathbb{Q}_p$

$\mathrm{Sel}_{\mathrm{str}} \overset{?}{=} 0$ (Leopoldt).

<u>Bloch-Kato conj</u>

For V coming from geom,

$\dim \mathrm{Sel}_{BK}(K,V) - \dim V^{G_K}$

$= \mathrm{ord}_{s=0} L(V^*(1), s)$.

Eg for $V =$ triv^l rep:

$-1 = -1$. ✓

$V = \mathbb{Q}_p(1)$: gives Dirichlet unit thm.

For $V_p(A)$: says

$$\mathrm{ord}_{s=1} L(A,s) = \mathrm{rk}\, A(K) + \mathrm{cork}\, \text{Ш}(p)$$

(close to BSD).

§1.4 Euler systems

$V \circlearrowleft G_{\mathbb{Q}}$

Can take $H^1(K, V)$ any K

If $L \supset K$ have corestriction

or norm maps

$$H^1(L, V) \to H^1(K, V).$$

$T \subset V$ $G_{\mathbb{Q}}$-stable $\mathcal{O}_E$-lattice

Σ fin set of primes including p & all ram. primes

<u>Def^n</u> An Euler system for (T, Σ) = collection $\underline{c} = (c_m)_{m \geq 1}$

$c_m \in H^1(\mathbb{Q}(\mu_m), T)$

st $\mathrm{norm}^{\mathbb{Q}(\mu_{m\ell})}_{\mathbb{Q}(\mu_m)}(c_{m\ell})$, ℓ prime,

$$= \begin{cases} c_m & \ell \in \Sigma \text{ or } \ell \mid m \\ P_\ell(V^*(1), \mathrm{Frob}_\ell^{-1}) \cdot c_m & \text{otherwise} \end{cases}$$

(in $\mathbb{Q}$ Gal$(\mathbb{Q}(\mu_m)/\mathbb{Q})$)

Why is this natural?

<u>Thm</u> (Rubin, Kolyvagin):

If $\exists$ E.S. for (T, Σ), and $c_1 \neq 0$,

+ technical cond^{n}s on T, then

$$\mathrm{Sel}_{\mathrm{strict}}(\mathbb{Q}, V^*(1)) = 0.$$

Key tool in work on BK conj.

Example of an ES :

"cyclotomic units."

$\chi : Gal(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \xrightarrow{\sim} \mathbb{Z}_p^\times$

$V = \mathbb{Q}_p(n)$. $\quad g \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}) : g \cdot v = \chi(g)^n \cdot v$

$W(n) = W \otimes \mathbb{Q}_p(n)$

$$u_m = 1 - \zeta_m \in \mathbb{Q}(\mu_m)^\times .$$

$$v_m = \mathrm{norm}^{\mathbb{Q}(\mu_{mp})}_{\mathbb{Q}(\mu_m)}(u_{mp})$$

$$c_m = K(v_m).$$

These are an ES for

$\{\mathbb{Z}_p(1), \{p\}\}$

<u>Note</u> : $\exists$ canonical bijection, for any $j \in \mathbb{Z}$,

$$(\text{ES for } (T, \Sigma)) \xrightarrow{\sim} (\text{ES for } (T(j), \Sigma)).$$

(Soulé twist).

$K/\mathbb{Q}$ Galois

$H^1(K, T)$ is a $\mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q})]$ mod.