An ES for Siegel modular forms
(joint w/ Loeffler + Skinner)

I genus 2 Siegel MFs

Def^n: $J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$, $G = GSp_4$

R ring,

$GSp_4(R) = \{(g, \nu) : g \in GL_4(R),\ \nu \in R^\times,$
s.t. $gJg^t = \nu J\}$

$Sp_4(R) = \{(g,\nu) \in GSp_4(R),\ \nu = 1\}$

$\mathcal{H}_2 = \{Z \in M_2(\mathbb{C}) : Z = \begin{pmatrix} y & z \\ x & y \end{pmatrix},$
$\mathrm{Im}\begin{pmatrix} x & y \\ y & z \end{pmatrix}$ pos. def$^{\text{nite}}\}$ $\circlearrowleft GSp_4(\mathbb{R})^+$

Def^n (Siegel 3-fold): Let $\mathcal{U} \subset G(\mathbb{A}_f)$
open, compact, suff. small

$\rightsquigarrow \tilde{Y}(\mathcal{U}) = GSp_4^+(\mathbb{Q}) \backslash G(\mathbb{A}_f) \times \mathcal{H}_2 / \mathcal{U}$

$=$ complex pts of smooth var $/\mathbb{Q}$

Examples:

i) $\mathcal{U} = \mathcal{U}_1(N)$

$$= \{(g,v) \in G(\hat{\mathbb{Z}}) : g \equiv \begin{pmatrix} * & * \\ 0_2 & I_2 \end{pmatrix} (N)\}$$

$\rightsquigarrow \tilde{Y}_1(N)$

ii) $\tilde{Y}_1(N) \times \mu_m = \tilde{Y}(\mathcal{U})$

$\mathcal{U} = \{(g,v) \in \mathcal{U}_1(N),\ g \equiv 1 (m)\}$

Def^n: $\tilde{\Gamma}_1(N) = U_1(N) \cap Sp_4(\mathbb{Z})$

A Siegel MF of genus 2, level N, wt (k,k) is a holo. fct.

$F : \mathcal{H}_2 \longrightarrow \mathbb{C}$

st. $F(g.Z) = \det(CZ+D)^k \, F(Z)$

$\forall g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \tilde{\Gamma}_1(N)$, $Z \in \mathcal{H}_2$

Fact: $\exists$ notion of cusp form

Def^n: Hecke algebra $= \{ U_1(N) \, g \, U_1(N),$
$g \in G(\mathbb{A}_f) \}$

$T(\ell) \longleftrightarrow \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ell & \\ & & & \ell \end{pmatrix}$

$T_1(\ell^2) \longleftrightarrow \begin{pmatrix} 1 & & & \\ & \ell & & \\ & & \ell^2 & \\ & & & \ell \end{pmatrix}$

$R(\ell) \longleftrightarrow \begin{pmatrix} \ell & & & \\ & \ell & & \\ & & \ell & \\ & & & \ell \end{pmatrix}$

Def^n: if F cusp form, eigenform for $T(\ell)$, $T_1(\ell^2)$, $R(\ell)$ with eval^s t, t_1, r, define

$$P_{spin,\ell}(F,X) = 1 - tX + \ell(t_1 + (\ell+1)r) X^2 - \ell^3 t r X^3 + \ell^6 r^2 X^4$$

Def^n: $L_{spin}(F,s) = \prod_\ell P_{spin}(F, \ell^{-s-\frac{3}{2}})^{-1}$

from now on, let F genus 2 cuspidal Siegel eigenform away from N of wt $(S,3)$ level N, assume: F non-endoscopic (= not a lift of auto form from a smaller gp)

Thm (Weissauer): $\exists\, E/\mathbb{Q}_p$ finite, + 4-dim^l gal. rep^n $V_{\mathcal{F}}/E$ st. $\forall \ell$

$\ell \nmid Np$,

$\det(1 - X\,\mathrm{Frob}_\ell^{-1} \mid V_{\mathcal{F}}) = P_{\mathrm{spin},\ell}(\mathcal{F}, X)$

$\mathcal{F}$ non-endoscopic $\Rightarrow$ $V_{\mathcal{F}}$ irred.

Aim: construct ES for $V_{\mathcal{F}}^*$

II Construction of ES

Step 1: find $V_{\mathcal{F}}^*$ in étale cohom. over $\overline{\mathbb{Q}}$

$\rightsquigarrow$ via HS spectral seq., determine where to construct ES in étale cohom $/\mathbb{Q}$ (num)

Step 2: construction of bottom class

Step 3: twisting element + classes over $\mathbb{Q}(\mu_m)$, $m \geq 1$

Step 4: norm rel^{n}s

Thm (W.): $\exists$ projection map

$\mathrm{Pr}_{\mathcal{F}}: H^3_{et}(\tilde{Y}_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Q}_p(3)) \otimes E \twoheadrightarrow V_{\mathcal{F}}^*$

HS spectral seq $\Rightarrow$ $\exists$ map

$H^4_{et}(\tilde{Y}_1(N)_{\mathbb{Q}(\mu_m)}, \mathbb{Q}_p(3))$

$\to H^1(\mathbb{Q}(\mu_m), H^3_{et}(\tilde{Y}_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Q}_p(3)))$

$\Rightarrow$ want to construct classes in $H^4_{et}(\tilde{Y}_1(N)_{\mathbb{Q}(\mu_m)}, \mathbb{Q}_p(n))$

some n satisfying norm rel^{ns}

$n=2$: Heegner pt style const^{n} (Cornut) $\rightsquigarrow$ anticyclotomic ES (embedding of codim 2 subvar.)

$n=4$: cupping together 4 units on $\tilde{Y}_1(N)$: noone knows

$n=3$: pushforward unit on codim 1 subvariety

$$\iota : GL_2 \times_{GL_1} GL_2 \hookrightarrow GSp_4$$

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right] \longmapsto \begin{pmatrix} a & & & b \\ & a' & b' & \\ & c' & d' & \\ c & & & d \end{pmatrix}$$

$$\rightsquigarrow Y_1(N) \times Y_1(N) \xhookrightarrow{\iota} \tilde{Y}_1(N)$$

$$\rightsquigarrow \iota_* : H^2_{et}(Y_1(N)^2, \mathbb{Q}_p(2))$$

$$\longrightarrow H^4_{et}(\tilde{Y}_1(N), \mathbb{Q}_p(3))$$

$$\iota_* : H^2_{\acute{e}t}(Y_1(N)^2, \mathbb{Q}_p(2)) \longrightarrow H^4_{\acute{e}t}(\tilde{Y}_1(N), \mathbb{Q}_p(3))$$

$$\begin{array}{ccc} \Psi & & \Psi \\ K_p(c\, g_{0,\frac{1}{N}}) \sqcup K_p(d\, g_{0,\frac{1}{N}}) & \longmapsto & {}_{c,d}\angle F_{1,N} \end{array}$$

Note: ${}_{c,d}\angle F_{1,N}$ is the image under $T_{\acute{e}t}$ of a motivic class

Thm (F. Lemma):

$$\left\langle r_{\mathbb{C}}\begin{pmatrix}\text{motivic}\\ \text{LF elt}\end{pmatrix}, \omega_{\mathcal{F}}\right\rangle \sim L'_{\text{spin}}(\mathcal{F}, -\tfrac{1}{2})$$

$\Rightarrow$ c.d $LF_{1,N}$ should be bottom class of ES

Step 3: classes $/\mathbb{Q}(\mu_m)$

Defn: $U(M,N) = \left\{(g,v) : g \equiv \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} \bmod \begin{pmatrix} M & M \\ N & N \end{pmatrix}\right\}$

Remark: $m \geq 1$

$\rightsquigarrow \exists\, s_m : \tilde{Y}(m, mN) \to \tilde{Y}_1(N) \times \mu_m$

now define classes $/\tilde{Y}(M,N)$, $M|N$

Lemma: if $M|N$, $M(M,N)$ is normalized by $u = \begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Def^n: $\iota_{M,N}: Y(M,N)^2 \xhookleftarrow{\iota} \tilde{Y}(M,N) \xrightarrow{u} \tilde{Y}(M,N)$

$\rightsquigarrow$

$$\rightsquigarrow (\iota_{M,N})_* : H^2_{et}(Y(M,N)^2, \mathbb{Q}_p(2)) \to H^4_{et}(\tilde{Y}(M,N), \mathbb{Q}_p(3))$$

$$\begin{array}{c} \cup \\ K_p({}_c g_{0,\frac{1}{N}}) \sqcup K_p({}_d g_{0,\frac{1}{N}}) \longmapsto {}_{c,d}\mathcal{L}\mathrm{Eis}_{M,N} \end{array}$$

Def^n: ${}_{c,d}\mathcal{L}F_{m,N} = (s_m)_* \, {}_{c,d}\mathcal{L}\mathrm{Eis}_{m,mN}$

Step 4: norm rel$^{\text{ns}}$

Prop$^{\text{n}}$: if $M \mid N$, $\ell \mid N$, $pr_*: \tilde{Y}(M, \ell N) \longrightarrow \tilde{Y}(M, N)$

$$\Rightarrow (pr_*)\left({}_{c,d}\angle F_{M,\ell N}\right) = {}_{c,d}\angle F_{M,N}$$

let $\tau_\ell : \tilde{Y}(\ell M, N) \to \tilde{Y}(M,N)$ given by right translation by

$$\begin{pmatrix} \ell & & & \\ & \ell & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$\leadsto$ factor ~~over~~ as

$$\tilde{Y}(\ell M, N) \longrightarrow \tilde{Y}(M(\ell), N) \xrightarrow{\tilde{\pi}_{2,\ell}} \tilde{Y}(M,N)$$

Thm: suppose $\ell | M$, $\ell M | N$

$\rightsquigarrow (\tau_\ell)_* ({}_{c,d}LEis_{\ell,M}) = \mathcal{U}'_\ell \cdot {}_{c,d}LEis_{M,N}$

$\mathcal{U}'_\ell$ = Hecke corresp. of $\tilde{Y}(M,N)$ given by elt of $G(\mathbb{A}_f)$ which is $\begin{pmatrix}\ell &&& \\ & \ell && \\ && 1 & \\ &&& 1\end{pmatrix}$ at ℓ, id elsewhere.

Cor: if $\ell | m$, $\ell m | N$,

$$\mathrm{norm}^{\mathbb{Q}(\mu_{m\ell})}_{\mathbb{Q}(\mu_m)} ({}_{cd}LF_{\ell m,N}) =$$

$$\mathcal{U}'_\ell \cdot {}_{cd}LF_{m,N}$$

Proof:

$$
\begin{tikzcd}
Y(\ell M, N)^2 \arrow[r, hook, "\iota_{M\ell, N}"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \tilde{Y}(\ell M, N) \arrow[d] \\
Y(\ell M, N)^2 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\Diamond"] & \tilde{Y}(M(\ell), N) \arrow[d] \\
Y(M, N)^2 \arrow[r, hook, "\iota_{M,N}"] & \tilde{Y}(M, N)
\end{tikzcd}
$$

$\tilde{\pi}_{2,\ell}$

τ_ℓ

h

$Y(M, N)$

$\square$

III Conclusion

Strategy works in great generality:

- $GL_2^3 \hookrightarrow GSp_6$ (Cauchi-Rodrigues)

  Siegel unit $\sqcup 1 \sqcup 1$

- $GU(1,1) \hookrightarrow GU(2,1)$ (LSZ)

  Siegel unit

- $GL_2 \times GL_2 \hookrightarrow GSp_4 \times GL_2$

  (project gp.)