p = odd prime, $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$

$\mathbb{Q}_\infty = \bigcup_{n \geq 0} \mathbb{Q}_n$, cyclot. $\mathbb{Z}_p$-ext. of $\mathbb{Q}$

$$\Lambda = \mathbb{Z}_p[[\Gamma]] = \varprojlim \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})]$$

$$\Lambda \cong \mathbb{Z}_p[[T]]$$

$$T \longleftrightarrow \gamma - 1$$

Let X be a f.g. Λ-module

$$X/TX = 0 \Longrightarrow X = 0$$

$$X_E(\mathbb{Q}_\infty) = \mathrm{Sel}_E(\mathbb{Q}_\infty)_p$$

$$\mathrm{Sel}_E(\mathbb{Q}_\infty)[T] = 0 \Longrightarrow \mathrm{Sel}_E(\mathbb{Q}_\infty) = 0$$

$$\mathrm{Sel}_E(\mathbb{Q}_\infty)[T] = \mathrm{Sel}_E(\mathbb{Q}_\infty)^\Gamma$$

Assume $E(\mathbb{Q})[p] = 0$

$$0 \to Sel_E(\mathbb{Q}) \to Sel_E(\mathbb{Q}_\infty)^{\Gamma}$$

$$\to \text{coker (rest)} \to 0$$

Assume E has good ordinary red at p

$X_E(\mathbb{Q}_\infty)$ has no nonzero finite Λ-submodules.

$Sel_E(\mathbb{Q}_\infty)$ is either 0 or infinite.

① $0 \to E(\mathbb{Q}_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_E(\mathbb{Q}_\infty)$

$\to$ Ш$_E(\mathbb{Q}_\infty)_p \to 0$

② $X_E(\mathbb{Q}_\infty)$ is a f.g.

torsion Λ-module

$\mathrm{Sel}_E(\mathbb{Q}_\infty) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\lambda_E} \times$ (a gp of bounded exponent)

λ_E

$\mu_E = 0$

$\iff \mathrm{Sel}_E(\mathbb{Q}_\infty)$ is divisible when is $Sel_E(\mathbb{Q}_\infty) \neq 0$?

$$0 \to Sel_E(\mathbb{Q}) \longrightarrow Sel_E(\mathbb{Q}_\infty)^\Gamma$$

$$\longrightarrow \text{coker} \to 0$$

$r = \text{rank}(E(\mathbb{Q}))$.

If $r \geq 1$, then

$$(\mathbb{Q}_p/\mathbb{Z}_p)^r \to Sel_E(\mathbb{Q})$$

$$\to (Sel_E(\mathbb{Q}_\infty))^\Gamma$$

$$\longrightarrow Sel_E(\mathbb{Q}_\infty)_{div}$$

$\lambda_E \geq r$

$r = 0$ & $Ш_E(\mathbb{Q})_p \neq 0$

$\mathrm{Sel}_E(\mathbb{Q}) = 0$

$\mathrm{Sel}_E(\mathbb{Q}_\infty) \neq 0$

$\iff p \mid c_\ell$ for some $\ell \mid N_E$, $\ell \neq p$

or

$$\alpha_p \equiv 1 \pmod p$$

$$|\tilde{E}(\mathbb{F}_p)| = 1 + p - a_p$$

$$= (1 - \alpha_p)(1 - \beta_p)$$

$\alpha_p \equiv 1 \pmod p \iff \tilde{E}(\mathbb{F}_p)[p] \neq 0$

$\iff a_p \equiv 1 \pmod p$

we then say p is anomalous for E

For action of $G_{\mathbb{Q}_p}$

$$0 \to \mathfrak{F}_E[p] \to E[p] \to \tilde{E}[p] \to 0$$

becomes

$$0 \to \mu_p \to E[p] \to \mathbb{Z}/p\mathbb{Z} \to 0$$

$$\Longrightarrow \text{Sel}_E(\mathbb{Q}_\infty) \text{ is infinite.}$$

$\mu_E > 0$

Suppose that we have an exact sequence

$$0 \to \mu_p \to E_1[p] \to \mathbb{Z}/p\mathbb{Z} \to 0$$

for action of $G_{\mathbb{Q}}$

$\Lambda/p^2\Lambda \qquad \Lambda/p\Lambda \times \Lambda/p\Lambda$

$\mu_E = 2$

Assume we have an exact sequence

$$0 \to \Phi \to E[p] \to \bar{\Phi} \to 0$$

where $\Phi \cong \mu_p$.

Let $\mathcal{F}_E$ = formal gp for E

$$\to \mathcal{F}_E(\bar{m}) \to E(\bar{\mathcal{O}}_p) \to \tilde{E}(\bar{\mathbb{F}}_p) \to 0$$

$\mathrm{ht}\ \mathcal{F}_E = 1$

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_E[p^\infty] \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & E[p^\infty] \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \tilde{E}[p^\infty] \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\
& \mathbb{Q}_p/\mathbb{Z}_p & (\mathbb{Q}_p/\mathbb{Z}_p)^2 & \mathbb{Q}_p/\mathbb{Z}_p &
\end{tikzcd}
$$

action of $G_{\mathbb{Q}_p}$ is unramified and Frob acts by a_p

$$\mathrm{Sel}_E(\Phi_{\mathbb{Q}}) = \left\{ [\sigma] \in \mathrm{Sel}_{rel}(\Phi_{\mathbb{Q}}) \;\middle|\; \begin{array}{l} \sigma \text{ is a 1-cocycle} \\ \text{with values} \\ \text{in } \mathcal{F}_E[p^\infty] \end{array} \right\}$$

Back to

$$0 \to \Phi \longrightarrow E[p^\infty] \to \Psi \to 0$$

(action for $G_{\mathbb{Q}}$)

$$\Phi \cong \mu_p$$

$$\Phi \subseteq \mathcal{F}_E[p] \subseteq \mathcal{F}_E[p^\infty]$$

$$\cancel{\mathrm{Sel}}$$

$$\mathrm{Sel}_E(\mathbb{Q}_\infty) = \ker\left(\mathrm{Sel}_{rel}(\mathbb{Q}_\infty) \to H^1\left(\mathbb{Q}_{p,\infty}, \frac{E[p^\infty]}{\mathcal{F}_E[p^\infty]}\right)\right)$$

$$H^1(\mathbb{Q}_{p,\infty}, \mathcal{F}_E[p^\infty]) \to H^1(\mathbb{Q}_{p,\infty}, E[p^\infty]) \to H^1\left(\mathbb{Q}_{p,\infty}, \frac{E[p^\infty]}{\mathcal{F}_E[p^\infty]}\right)$$

~~$H^1_{rel}(\Phi_{\infty}, \Phi) \hookrightarrow H^1_{rel}(\Phi_{\infty}, E[p^\infty]$~~

$$H^1_{rel}(\Phi_{\infty}, \Phi) \hookrightarrow H^1_{rel}(\Phi_{\infty}, E[p^\infty])$$

$$\| $$

$$\mathrm{Sel}_{\underline{rel}}(\Phi_{\infty})$$

finite kernel

$$\cancel{U_{\infty} \otimes \Phi_p} \longrightarrow H^1(\Phi_{\infty}, \Phi)$$

$$U_{\infty}/U^p_{\infty} \longrightarrow H^1_{rel}(\Phi_{\infty}, \Phi$$