Project.

Let K be an imaginary quadratic field, and p a rational prime which splits in K into two distinct primes $\mathfrak{p}, \mathfrak{p}^*$. By class field theory, there is a unique $\mathbb{Z}_p$-extension K_∞/K which is unramified outside of $\mathfrak{p}$. Assume now that F is an arbitrary finite extension of K. We call

$$F_\infty = FK_\infty$$

the "split prime" $\mathbb{Z}_p$-extension of F. It seems probable that this split prime $\mathbb{Z}_p$-extension F_∞/F has many properties in close analogy with those of the cyclotomic $\mathbb{Z}_p$-extension of F. The aim of the project is to discuss several of these analogies, and establish a few rather limited theoretical and numerical examples in support of them.

Part I

Analogues of the Leopoldt and weak Leopoldt conjectures.

We assume from now on that K is an imaginary quadratic in which p splits into $\mathfrak{p}, \mathfrak{p}^*$, and that F is an arbitrary finite extension of K. For each prime v of F lying above $\mathfrak{p}$, write U_v for the group of local units in the completion of F at v which are $\equiv 1 \bmod v$. Put $U_F = \prod_{v|\mathfrak{p}} U_v$. Thus U_F is a $\mathbb{Z}_p$-module of rank equal to r_2, where r_2 denotes the number of complex primes of F $(= [F:K])$. Let E_F be the group of all global units of F which are $\equiv 1 \bmod v$ for all $v|\mathfrak{p}$. By Dirichlet's theorem, E_F has $\mathbb{Z}$-rank equal to $r_2 - 1$. Now we have the obvious embedding of E_F into U_F, and we define $\overline{E_F}$ to be the closure of the image in U_F under the $\mathfrak{p}$-adic topology (equivalently, $\overline{E_F}$ is the $\mathbb{Z}_p$-submodule of U_F

which is generated by the image of E_F). Thus $\overline{E}_F$ must have $\mathbb{Z}_p$-rank equal to $r_2 - 1 - \delta_{F,\mathfrak{p}}$ for some integer $\delta_{F,\mathfrak{p}} \geqslant 0$.

<u>$\mathfrak{p}$-adic Leopoldt conjecture</u>. $\delta_{F,\mathfrak{p}} = 0$.

Again global class field theory gives a Galois-theoretic interpretation of this conjecture. Let L be the p-Hilbert class field of F, and let M be the maximal abelian p-extension of F, which is unramified outside the set of primes of F lying above $\mathfrak{p}$. Then the Artin map induces an isomorphism

$$U_F / \overline{E}_F \xrightarrow{\sim} \mathrm{Gal}(M/L),$$

whence we obtain:–

<u>Theorem</u> 1.1. Let M be the maximal abelian p-extension of F which is unramified outside the primes of F lying above $\mathfrak{p}$. Then $\mathrm{Gal}(M/F)$ is a finitely generated $\mathbb{Z}_p$-module of $\mathbb{Z}_p$-rank equal to $1 + \delta_{F,\mathfrak{p}}$.

<u>Corollary 1.2</u>. $\delta_{F,\mathfrak{p}} = 0$ if and only if $\mathrm{Gal}(M/F_\infty)$ is finite.

Let $\sigma_1, \dots, \sigma_{r_2}$ be the embeddings of F into $\overline{\mathbb{Q}}_p$ extending the embedding of K into $\mathbb{Q}_p$ given by $\mathfrak{p}$. Let $\varepsilon_1, \dots, \varepsilon_{r_2 - 1}$ be a $\mathbb{Z}$-basis of E_F modulo torsion. Note that the series $\log x$ converges on all principal units of $\overline{\mathbb{Q}}_p$. Define

$$R_{\mathfrak{p}}(F) = \det\left(\log \sigma_i(\varepsilon_j)\right)_{i,j = 1, \dots, r_2 - 1}.$$

<u>Ex</u> 1.1. Prove that $\delta_{F,\mathfrak{p}} \neq 0$ if and only if $R_{\mathfrak{p}}(F) \neq 0$. If F is an abelian extension of K, use Baker's theorem that $\log \varepsilon_1, \dots, \log \varepsilon_{r_2 - 1}$ are linearly independent over the field of algebraic numbers to show that $R_{\mathfrak{p}}(F) \neq 0$.

Exc 1.2. With the help of SAGE or MAGMA, one can often check numerically that $R_{\mathfrak{p}}(F) \neq 0$ even when F is not an abelian extension of K. Here is one example. Take $K = \mathbb{Q}(i)$, $p = 5$, and $\mathfrak{p} = (1-2i)\mathbb{Z}[i]$. Let $w = \frac{1-\sqrt{5}}{2}$, and take

$$F = K(w, \beta^{1/4}), \text{ where } \beta = w(1-2i)^3.$$

Show that $F = \mathbb{Q}(\delta)$, where δ is a root of $x^8+4x^6+9x^4+10x^2+5=0$. Using one of the above programmes, find the group of global units of F, and check that $\mathrm{ord}_{\mathfrak{p}}(R_{\mathfrak{p}}(F)) = 3/2$.

We now turn to the weak $\mathfrak{p}$-adic Leopoldt conjecture for F_∞/F. For each $n \geqslant 0$, let F_n be the unique extension of F contained in F_∞ with $[F_n : F] = p^n$. Let $\delta_{F_n,\mathfrak{p}}$ denote the $\mathfrak{p}$-adic default of Leopoldt for F_n.

<u>Weak $\mathfrak{p}$-adic Leopoldt conjecture for F_∞/F.</u>

$$\delta_{F_n,\mathfrak{p}} \text{ is bounded as } n \to \infty.$$

Of course, the analogue of this statement for the cyclotomic $\mathbb{Z}_p$-extension of F was proven by Iwasawa, but unfortunately his proof does not seem to extend to F_∞/F.

There is an equivalent formulation of this conjecture purely in terms of an Iwasawa module. Let $M(F_\infty)$ be the maximal abelian p-extension of F_∞, which is unramified outside the set of primes of F_∞ lying above $\mathfrak{p}$, and put

$$X(F_\infty) = \mathrm{Gal}(M(F_\infty)/F_\infty).$$

Clearly $M(F_\infty)$ is Galois over F, and so $\Gamma = \mathrm{Gal}(F_\infty/F)$ acts on $X(F_\infty)$ in the usual fashion. It follows that $X(F_\infty)$ is a module over the Iwasawa algebra $\Lambda(\Gamma)$ of Γ, and it is easily seen to be finitely generated over $\Lambda(\Gamma)$. Moreover, we have

$$(X(F_\infty))_{\Gamma_n} = \mathrm{Gal}(M_n/F_\infty),$$

where M_n is the maximal abelian p-extension of F_n, which is unramified outside the primes of F_n lying above $\mathfrak{p}$.

Theorem 1.3. $X(F_\infty)$ is $\Lambda(\Gamma)$-torsion if and only if $\delta_{F_n,\mathfrak{p}}$ is bounded as $n \to \infty$.

Corollary 1.4 If $\delta_{F,\mathfrak{p}} = 0$, then $\delta_{F_n,\mathfrak{p}}$ is bounded as $n \to \infty$.

Of course, one can use Corollary 1.4 to prove the weak $\mathfrak{p}$-adic Leopoldt conjecture in numerical examples (e.g. in the example of Ex 1.2).

There are two other important aspects of the weak $\mathfrak{p}$-adic Leopoldt conjecture for F_∞/F which we mention briefly. Firstly, there is an exact formula for $\#(\mathrm{Gal}(M/F_\infty))$ when $R_\mathfrak{p}(F) \neq 0$, which is a first hint that there may be a "main conjecture" for $X(F_\infty)$. Let $h(F)$ be the class number of F, $w(F)$ the number of roots of unity in F, and $\Delta(F/K)$ any generator of the discriminant ideal of F/K. If v is a finite place of F, Nv will denote the cardinality of the residue field of v.

Ex 1.3 (see [CW 1]). Assume that $R_\mathfrak{p}(F) \neq 0$. Then

$$[M : F_\infty] = \left| \frac{p^{e(F)+1} h(F) R_\mathfrak{p}(F)}{w(F)\sqrt{\Delta(F/K)}} \prod_{v \mid \mathfrak{p}} \left(1 - \frac{1}{Nv}\right) \right|_p^{-1},$$

where the integer $e(F)$ is defined by $F \cap K_\infty = K_{e(F)}$.

Here the p-adic valuation on $\overline{\mathbb{Q}}_p$ is normalized by $\left|\frac{1}{p}\right|_p = p$.

Secondly, the weak Leopoldt conjecture for F_∞/F is closely related to the Iwasawa theory for F_∞/F of elliptic curves with complex multiplication by the full ring of integers of K (see [C 2]).

Ex 1.4. In the specific numerical example discussed in Example 1.2, use the formula of Example 1.3 to prove that $X(F_\infty) = 0$.

Part II

Analogue of Iwasawa's $\mu = 0$ conjecture.

We first recall Iwasawa's $\mu = 0$ conjecture. Let F be any finite extension of $\mathbb{Q}$, p any prime number, and F_∞^{cyc}/F the cyclotomic $\mathbb{Z}_p$-extension of F.

Iwasawa's $\mu = 0$ conjecture. Let L_∞ be the maximal abelian p-extension of F_∞^{cyc}, which is unramified everywhere. Then $\mathrm{Gal}(L_\infty/F_\infty^{cyc})$ is a finitely generated $\mathbb{Z}_p$-module.

Iwasawa (see [IW2]) has given examples of fields F and primes p for which the analogue of this conjecture is false for certain non-cyclotomic $\mathbb{Z}_p$-extensions of F. The best result to date in support of Iwasawa's conjecture is the following:—

Theorem (Ferrero - Washington, Sinnott). Iwasawa's $\mu = 0$ conjecture is valid for all finite abelian extensions F of $\mathbb{Q}$, and all primes p.

Now assume F is a finite extension of an imaginary quadratic field K, and F_∞/F is the split prime $\mathbb{Z}_p$-extension of K. Let $\mathfrak{p}$ be the degree 1 prime of K giving rise to F_∞/F.

Split prime analogue of Iwasawa's $\mu = 0$ conjecture. Let F_∞/F be the split prime $\mathbb{Z}_p$-extension, and let $M(F_\infty)$ be the maximal abelian p-extension of F_∞ which is unramified outside the primes of F_∞ lying above $\mathfrak{p}$. Then $X(F_\infty) = \mathrm{Gal}(M(F_\infty)/F_\infty)$ is a finitely generated $\mathbb{Z}_p$-module.

Ex 1.5. Let J/F be a finite Galois extension, whose Galois group is cyclic of order p. Let J_∞/J and F_∞/F be the split prime $\mathbb{Z}_p$-extensions, so that $J_\infty = F_\infty J$. If $X(F_\infty)$ is a finitely generated $\mathbb{Z}_p$-module, prove that $X(J_\infty)$ is a finitely generated $\mathbb{Z}_p$-module.

There is a considerable body of literature showing how Sinnott's beautiful proof, by analytic means, of the Iwasawa $\mu = 0$ for conjecture for the cyclotomic $\mathbb{Z}_p$-extension of a finite abelian extension F of $\mathbb{Q}$ can be generalized to the split prime $\mathbb{Z}_p$-extension $\mu = 0$ conjecture given above for F a finite abelian extension of K. However, it must be said that the analytic arguments needed in the split prime case are not fully worked out in the existing literature, especially for the primes $p = 2, 3$. There would be real interest in writing a fully detailed and comprehensible proof showing that Sinnott's method proves in complete generality the above $\mu = 0$ conjecture for the split prime $\mathbb{Z}_p$-extension of any finite abelian extension F of K.