

L4

$[F: \mathbb{Q}] < \infty$, F_∞/F - arbitrary

\mathbb{Z}_p - extension, $\Gamma = \text{Gal}(F_\infty/F)$.

$A'_n = p$ -primary subgroup of I'_n/P'_n

$n \leq m$

$$i_{n,m}: A'_n \rightarrow A'_m$$

$$N_{m,n}: A'_m \rightarrow A'_n$$

Define Γ -modules

$$A'_\infty = \varinjlim A'_n \quad - \text{discrete}$$

$$W'_\infty = \varprojlim A'_n \quad - \text{compact}$$

2.

Theorem A. $\text{Hom}(A'_{\infty}, \mu_{p^{\infty}})$ is a torsion $\Lambda(\Gamma)$ -module.

As usual, it suffices to prove

$$Z'_{\infty} = \text{Hom}(A'_{\infty}, \mathcal{O}_p/\mathbb{Z}_p)$$

is $\Lambda(\Gamma)$ -torsion, since

$$\text{Hom}(A'_{\infty}, \mu_{p^{\infty}}) = \text{Hom}(A'_{\infty}, \mathcal{O}_p/\mathbb{Z}_p) \otimes_{T_p(\mu)}$$

Lemma: X a f.g. $\Lambda(\Gamma)$ -module.

X is $\Lambda(\Gamma)$ -torsion $\iff X \otimes_{T_p(\mu)}$
is $\Lambda(\Gamma)$ -torsion.

3.

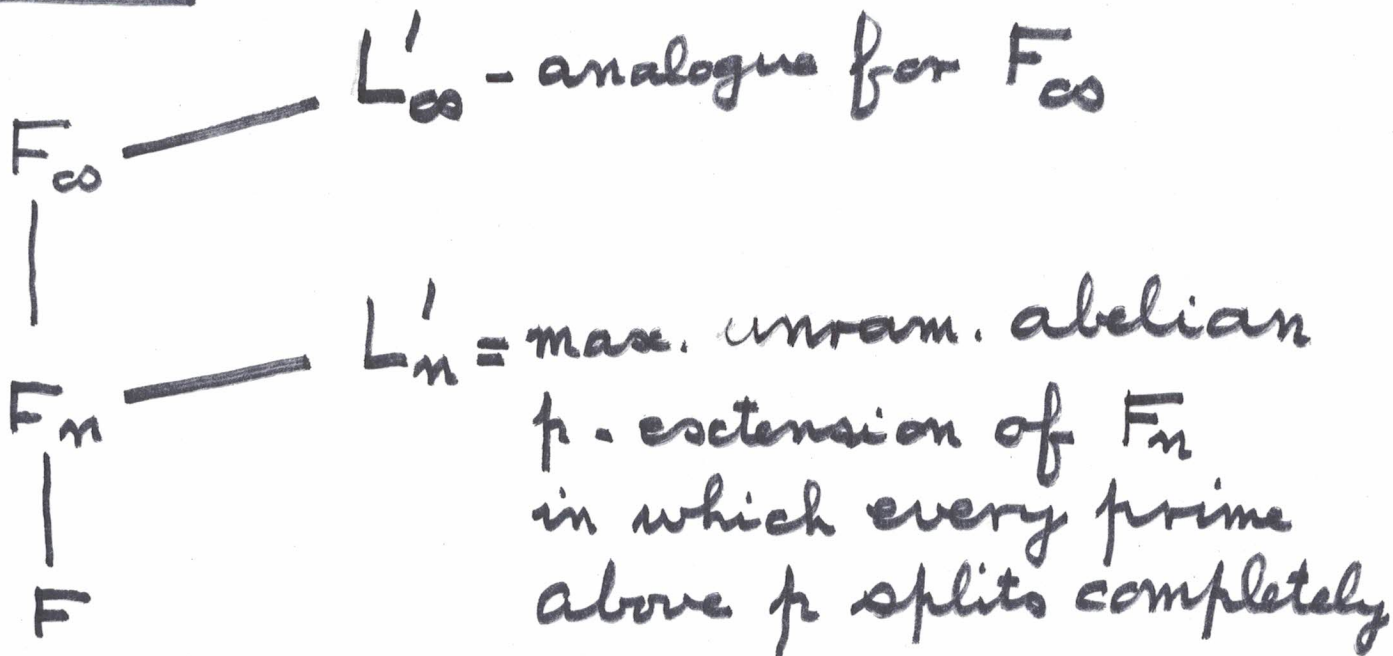
Strategy.

Part I. Prove $W'_\infty = \varprojlim A'_n$ is a finitely generated torsion $\Lambda(\Gamma)$ -module.

Part II. Relate the two $\Lambda(\Gamma)$ -modules

$W'_\infty = \varprojlim A'_n$ and $\text{Hom}(A'_\infty, \mathbb{Q}_p/\mathbb{Z}_p)$.

Part I.



4.

Prop. $W_\infty \simeq \text{Gal}(L'_\infty/F_\infty)$.

Obvious $L'_\infty = \bigcup_{n \geq 0} L'_n$.

Artin map for all $n \geq 0$ gives an isomorphism

$$A_n' \simeq \text{Gal}(L'_n/F_n).$$

What happens as we vary n ?

n_0 : All ramified primes in F_∞/F are totally ramified in F_∞/F_{n_0} .

$$n \geq n_0 \quad m \geq n \geq n_0$$

$$L'_n \cap F_m = F_n$$

$$\text{Gal}(L'_n F_m/F_n) \simeq \text{Gal}(L'_n/F_n)$$

5.
Global class field theory \Rightarrow

$$\begin{array}{ccc}
 A'_m & \xrightarrow{\sim} & \text{Gal}(L'_m/F_m) \\
 N_{m,n} \downarrow & & \downarrow \\
 A'_n & \xrightarrow{\sim} & \text{Gal}(L'_n F_m/F_m) = \text{Gal}(L'_n/F_m)
 \end{array}$$

Pass to projective limit.

Hence to prove part I we have to show $\text{Gal}(L'_\infty/F_\infty)$ is a f.g. torsion $\wedge(\Gamma)$ -module.

$$\begin{array}{ccc}
 & L'_n & L'_\infty \\
 F_\infty & \text{---} & \\
 | & \text{abelian} & \\
 F_n & & \\
 | & & \\
 F & &
 \end{array}$$

$L'_n = \text{max. abelian extension of } F_n \text{ in } L'_\infty.$

6.

Key trivial remark.

$$\mathcal{L}'_n \supset F_\infty \text{ and so } \mathcal{L}'_n \neq L'_n.$$

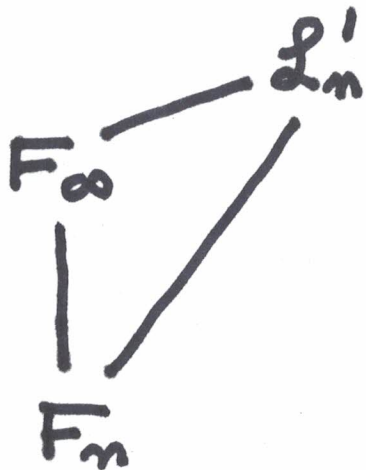
Not true in general that $\mathcal{L}'_n = L'_n F_\infty$

$$W'_\infty = \text{Gal}(L'_\infty / F_\infty).$$

Hence, as always, we have

$$(W'_\infty)_{\Gamma_n} = \text{Gal}(\mathcal{L}'_n / F_\infty)$$

Assume $n \geq n_0$: s = number of ram. primes in F_∞ / F_{n_0}



F_∞ / F_n : s totally ramified primes

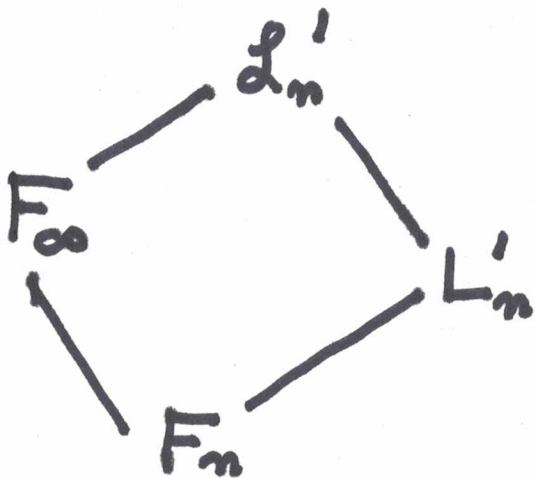
$\mathcal{L}'_n / F_\infty$: all primes above p split completely

7.

$\omega_1, \dots, \omega_s$ primes of F_n which ramify in L_n'

T_1, \dots, T_s inertial subgroups in $\text{Gal}(L_n'/F_n)$

$$T_i \cong \Gamma_n \cong \mathbb{Z}_p$$



$L_n' = \text{max. unramified extension of } F_n \text{ in } L_n'$

$$\Rightarrow \text{Gal}(L_n'/L_n) = T_1 \dots T_s$$

$$\Rightarrow \mathbb{Z}_p\text{-rank of Gal}(L_n'/L_n) \leq s$$

$$\Rightarrow \mathbb{Z}_p\text{-rank of Gal}(L_n'/F_n) \leq s$$

$$\Rightarrow \mathbb{Z}_p\text{-rank of Gal}(L_n'/F_\infty) \leq s-1$$

Hence we have proven:-

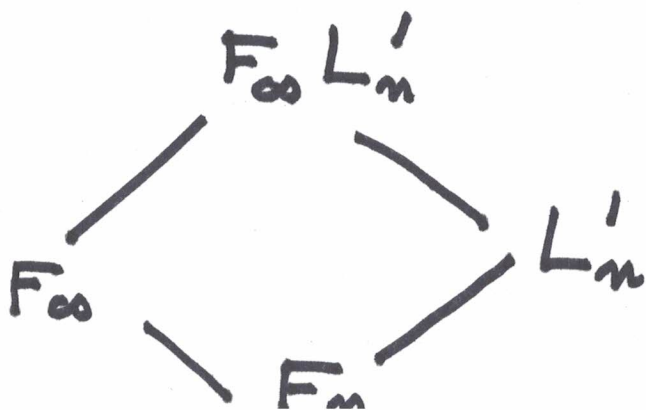
Theorem. \mathbb{Z}_p -rank of $(W_\infty')_{\Gamma_n} \leq s-1$
for all $n \geq n_0$.

Corollary. W_∞' is $\Lambda(\Gamma)$ -torsion.

Part II. $Z_\infty' = \text{Hom}(A_\infty', \mathbb{Q}_p/\mathbb{Z}_p)$

We want to determine relation
between the $\Lambda(\Gamma)$ -modules W_∞'
and Z_∞' to deduce Z_∞' is
 $\Lambda(\Gamma)$ -torsion.

$$n \geq n_0 \quad F_\infty \cap L_n' = F_n$$



9.

$$A'_n \simeq \text{Gal}(F_\infty L'_n / F_\infty) \quad n \geq n_0.$$

How do we handle the fact that we only have good behaviour for $n \geq n_0$?

Introduce : $V'_\infty = \text{Gal}(L'_\infty / L'_{n_0} F_\infty)$

Thus V'_∞ is of finite index in W'_∞ .

$$\Lambda(\Gamma) \simeq \mathbb{Z}_p[\Gamma] \quad \gamma \mapsto 1 + T$$

$$\begin{aligned} \omega_n &= (1 + T)^{p^n} - 1 \quad n \geq 0 \\ &= \gamma^{p^n} - 1 \end{aligned}$$

$$(M)_{\Gamma_n} = M / \omega_n M$$

Defn. $n \geq n_0$, $v_{n_0, n} = \omega_n / \omega_{n_0}$

Lemma. For $n \geq n_0$,

$$L'_n = L'_n \not\cong L'_{n_0}$$

→

$$\text{Gal}(L'_\infty / L'_n F_\infty) = \mathcal{V}_{n_0, n} V'_\infty.$$

⇒

$$\bullet A'_n \cong W'_\infty / \mathcal{V}_{n_0, n} V'_\infty.$$

Also if $m \geq n \geq n_0$,

$$\begin{array}{ccc} A'_n & \cong & W'_\infty / \mathcal{V}_{n_0, n} V'_\infty \\ \downarrow & & \downarrow \times \mathcal{V}_{n, m} \\ A'_m & \cong & W'_\infty / \mathcal{V}_{n_0, m} V'_\infty \end{array}$$

Hence

$$A'_{\infty} = \varinjlim A'_n = \varinjlim_{n \geq n_0} W'_{\infty} / \nu_{n_0, n} V'_{\infty}$$

But $V'_{\infty} \subset W'_{\infty}$ of finite index \Rightarrow

$$\varinjlim_{n \geq n_0} W'_{\infty} / \nu_{n_0, n} V'_{\infty} = \varinjlim_{n \geq n_0} V'_{\infty} / \nu_{n_0, n} V'_{\infty}.$$

Theorem.

$$Z'_{\infty} = \text{Hom}(A'_{\infty}, \mathcal{O}_T / \mathbb{Z}_T)$$

$$\cong \varprojlim \text{Hom}(V'_{\infty} / \pi_n V'_{\infty}, \mathcal{O}_T / \mathbb{Z}_T)$$

$$\pi_n = \nu_{n_0, n_0+n}.$$

Algebra. \times any f.g. torsion $\Lambda(\Gamma)$ -module

$X/\pi_n X$ finite for all $n \geq 0$

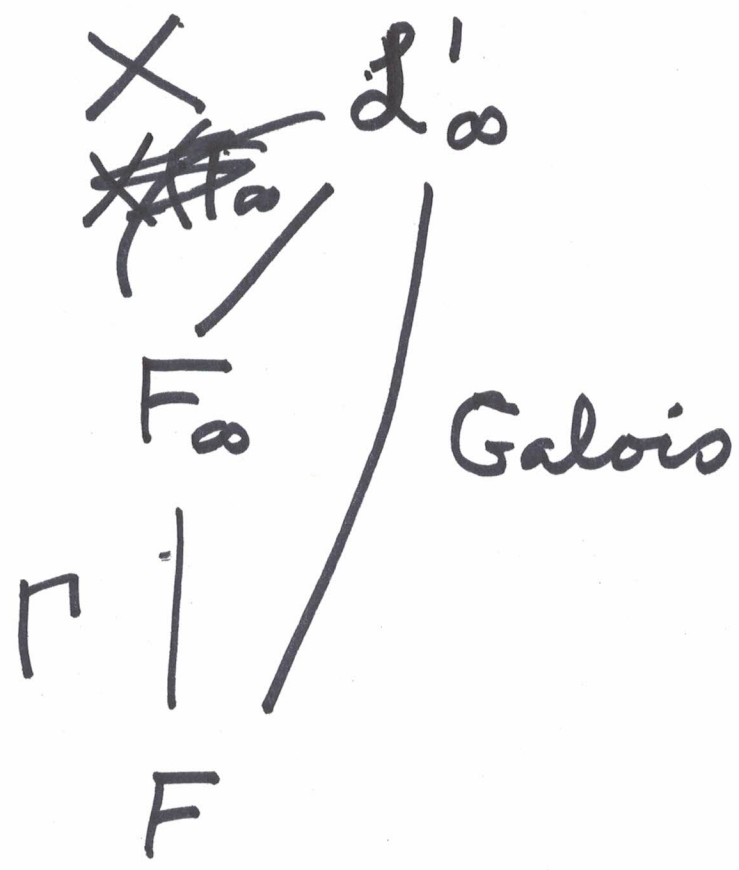
Then

$$\begin{aligned} & \varprojlim \text{Hom}(X/\pi_n X, \mathbb{Q}_p/\mathbb{Z}_p) \\ &= \alpha(X) = \text{Ext}_{\Lambda(\Gamma)}^1(X, \Lambda(\Gamma)). \end{aligned}$$

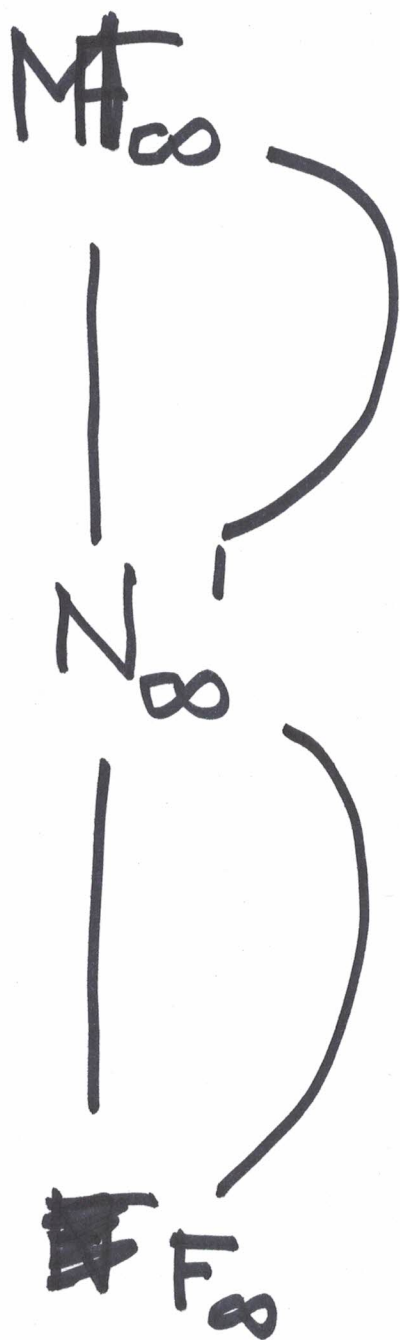
Conclusion. Z_{∞}' is a f.g.

torsion $\Lambda(\Gamma)$ -module, with
no non-zero finite $\Lambda(\Gamma)$ -submodule

Theorem A is proven!



$$(X)_{\Gamma} = \text{Gal}(L'/F_0)$$



No non-zero
finite $\Lambda(\Gamma)$
submodule

$\text{Gal}(N_\infty/F_\infty)$ has
no \mathbb{Z}_p -torsion

$\Rightarrow \text{Gal}(M_\infty/F_\infty)$ has
no non-zero finite
 $\Lambda(\Gamma)$ -submodule.

