L4

$[F:\mathbb{Q}] < \infty$, F_∞/F - arbitrary

$\mathbb{Z}_p$ - extension, $\Gamma = \mathrm{Gal}(F_\infty/F)$.

A'_n = p-primary subgroup of I'_n/P'_n

$n \leq m$

$i_{n,m} : A'_n \longrightarrow A'_m$

$N_{m,n} : A'_m \longrightarrow A'_n$

Define Γ-modules

$A'_\infty = \varinjlim A'_n$ - discrete

$W'_\infty = \varprojlim A'_n$ - compact

<u>Theorem A</u>. $\mathrm{Hom}(A'_\infty, \mu_{p^\infty})$ is a torsion $\Lambda(\Gamma)$-module.

As usual, it suffices to prove

$$Z'_\infty = \mathrm{Hom}(A'_\infty, \mathbb{Q}_p/\mathbb{Z}_p)$$

is $\Lambda(\Gamma)$-torsion, since

$$\mathrm{Hom}(A'_\infty, \mu_{p^\infty}) = \mathrm{Hom}(A'_\infty, \mathbb{Q}_p/\mathbb{Z}_p) \otimes T_p(\mu)$$

<u>Algebra</u>: X a f.g. $\Lambda(\Gamma)$-module. X is $\Lambda(\Gamma)$-torsion $\Leftrightarrow$ $X \otimes T_p(\mu)$ is $\Lambda(\Gamma)$-torsion.

Strategy.

Part I. Prove $W'_\infty = \varprojlim A'_n$ is a finitely generated torsion $\Lambda(\Gamma)$-module.

Part II. Relate the two $\Lambda(\Gamma)$-modules

$$W'_\infty = \varprojlim A'_n \text{ and } \mathrm{Hom}(A'_\infty, \mathbb{Q}_p/\mathbb{Z}_p).$$

Part I.

F_∞ — L'_∞ - analogue for F_∞

$|$

F_n — L'_n = max. unram. abelian p-extension of F_n in which every prime above p splits completely

$|$

F

<u>Prop.</u> $W'_\infty \xrightarrow{\sim} \mathrm{Gal}(L'_\infty/F_\infty)$.

Obvious $L'_\infty = \bigcup_{n \geq 0} L'_n$.

Artin map for all $n \geq 0$ gives an isomorphism

$$A'_n \xrightarrow{\sim} \mathrm{Gal}(L'_n/F_n).$$

What happens as we vary n?

<u>n_0</u> : All ramified primes in F_∞/F are totally ramified in F_∞/F_{n_0}.

$n \geq n_0 \qquad m \geq n \geq n_0$

$$L'_n \cap F_m = F_n$$

$$\mathrm{Gal}(L'_n F_m/F_n) \simeq \mathrm{Gal}(L'_n/F_n)$$

Global class field theory $\Rightarrow$

$$\begin{tikzcd}
A'_m \arrow[r, "\sim"] \arrow[d, "N_{m,n}"'] & \mathrm{Gal}(L'_m/F_m) \arrow[d, twoheadrightarrow] \\
A'_n \arrow[r, "\sim"] & \mathrm{Gal}(L'_n F_m/F_m) = \mathrm{Gal}(L'_n/F_m)
\end{tikzcd}$$

Pass to <u>projective limit</u>.

Hence to prove part I we have to show $\mathrm{Gal}(L'_\infty/F_\infty)$ is a f.g. torsion $\Lambda(\Gamma)$-module.

$$\begin{tikzcd}[column sep=small]
 & & L'_\infty \\
 & \mathcal{L}'_n \arrow[ur, dash] & \\
F_\infty \arrow[ur, dash] & & \\
F_n \arrow[u, dash] \arrow[uur, dash, bend right, "\text{abelian}"'] & & \\
F \arrow[u, dash] & &
\end{tikzcd}$$

$\mathcal{L}'_n$ = max. abelian extension of F_n in L'_∞.

<u>Key trivial remark</u>.

$\mathcal{L}_n' \supset F_\infty$ and so $\mathcal{L}_n' \neq L_n'$.

<u>Not</u> true in general that $\mathcal{L}_n' = L_n' F_\infty$

$W_\infty' = \mathrm{Gal}(L_\infty'/F_\infty)$.

Hence, as always, we have

$$(W_\infty')_{\Gamma_n} = \mathrm{Gal}(\mathcal{L}_n'/F_\infty)$$

Assume $n \geqslant n_0$: s = number of ram. primes in F_∞/F_{n_0}

$$\begin{tikzcd} & \mathcal{L}_n' \\ F_\infty \arrow[ur, dash] & \\ F_n \arrow[u, dash] \arrow[uur, dash] & \end{tikzcd}$$

F_∞/F_n : s totally ramified primes

$\mathcal{L}_n'/F_\infty$: all primes above p split completely

$\omega_1, \dots, \omega_s$ primes of F_n which ramify in $\mathcal{L}_n'$

$T_1, \dots, T_s$ inertial subgroups in $\mathrm{Gal}(\mathcal{L}_n'/F_n)$

$T_i \xrightarrow{\sim} \Gamma_n \xrightarrow{\sim} \mathbb{Z}_p$

$$\begin{tikzcd}[column sep=small]
 & \mathcal{L}_n' \arrow[dl, dash] \arrow[dr, dash] & \\
F_\infty \arrow[dr, dash] & & L_n' \arrow[dl, dash] \\
 & F_n &
\end{tikzcd}$$

L_n' = max. unramified extension of F_n in $\mathcal{L}_n'$.

$\Rightarrow \mathrm{Gal}(\mathcal{L}_n'/L_n') = T_1 \dots T_s$

$\Rightarrow \mathbb{Z}_p$-rank of $\mathrm{Gal}(\mathcal{L}_n'/L_n') \leq s$

$\Rightarrow \mathbb{Z}_p$-rank of $\mathrm{Gal}(\mathcal{L}_n'/F_n) \leq s$

$\Rightarrow \mathbb{Z}_p$-rank of $\mathrm{Gal}(\mathcal{L}_n'/F_\infty) \leq s-1$

Hence we have proven:–

<u>Theorem</u>. $\mathbb{Z}_p$-rank of $(W'_\infty)_{\Gamma_n} \leqslant s-1$ for all $n \geqslant n_0$.

<u>Corollary</u>. W'_∞ is $\Lambda(\Gamma)$-torsion.

<u>Part II</u>. $Z'_\infty = \mathrm{Hom}(A'_\infty, \mathbb{Q}_p/\mathbb{Z}_p)$

We want to determine relation between the $\Lambda(\Gamma)$-modules W'_∞ and Z'_∞ to deduce Z'_∞ is $\Lambda(\Gamma)$-torsion.

$n \geqslant n_0 \qquad F_\infty \cap L'_n = F_n$

$$\begin{tikzcd}
 & F_\infty L'_n \arrow[dl, dash] \arrow[dr, dash] & \\
F_\infty \arrow[dr, dash] & & L'_n \arrow[dl, dash] \\
 & F_n &
\end{tikzcd}$$

$$A'_n \xrightarrow{\sim} \mathrm{Gal}(F_\infty L'_n / F_\infty) \quad n \geq n_0.$$

How do we handle the fact that we only have good behaviour for $n \geq n_0$?

<u>Introduce</u> : $V'_\infty = \mathrm{Gal}(L'_\infty / L'_{n_0} F_\infty)$

Thus V'_∞ is of finite index in W'_∞.

$$\Lambda(\Gamma) \xrightarrow{\sim} \mathbb{Z}_p[[T]] \quad \gamma \longmapsto 1+T$$

$$\begin{aligned} \omega_n &= (1+T)^{p^n} - 1 \quad n \geq 0 \\ &= \gamma^{p^n} - 1 \end{aligned}$$

$$(M)_{\Gamma_n} = M/\omega_n M$$

<u>Defn</u>. $n \geq n_0$, $\nu_{n_0,n} = \omega_n / \omega_{n_0}$

<u>Lemma</u>. For $n \geqslant n_0$,

$$\mathcal{L}'_n = L'_n \cancel{F_\infty} \mathcal{L}'_{n_0}$$

$\Longrightarrow$

$$\mathrm{Gal}\left(L'_\infty / L'_n F_\infty\right) = \nu_{n_0,n} V'_\infty .$$

$\Longrightarrow$

$$A'_n \xrightarrow{\sim} W'_\infty / \nu_{n_0,n} V'_\infty .$$

Also if $m \geqslant n \geqslant n_0$,

$$\begin{tikzcd}
A'_n \arrow[r, "\sim"] \arrow[d] & W'_\infty / \nu_{n_0,n} V'_\infty \arrow[d, "\times \nu_{n,m}"] \\
A'_m \arrow[r, "\sim"] & W'_\infty / \nu_{n_0,m} V'_\infty
\end{tikzcd}$$

Hence

$$A'_\infty = \varinjlim A'_n = \varinjlim_{n \geq n_0} W'_\infty / \nu_{n_0,n} V'_\infty$$

But $V'_\infty \subset W'_\infty$ of finite index $\Rightarrow$

$$\varinjlim_{n \geq n_0} W'_\infty/\nu_{n_0,n} V'_\infty = \varinjlim V'_\infty/\nu_{n_0,n} V'_\infty.$$

<u>Theorem</u>.

$$Z'_\infty = \mathrm{Hom}(A'_\infty, \mathbb{Q}_p/\mathbb{Z}_p)$$

$$\xrightarrow{\sim} \varprojlim \mathrm{Hom}(V'_\infty/\pi_n V'_\infty, \mathbb{Q}_p/\mathbb{Z}_p)$$

$\pi_n = \nu_{n_0, n_0+n}$.

<u>Algebra</u>. X any f.g. torsion $\Lambda(\Gamma)$-module

$X/\pi_n X$ finite for all $n \geqslant 0$

Then

$$\varprojlim \mathrm{Hom}(X/\pi_n X, \mathbb{Q}_p/\mathbb{Z}_p)$$

$$= \alpha(X) = \mathrm{Ext}^1_{\Lambda(\Gamma)}(X, \Lambda(\Gamma)).$$

<u>Conclusion</u>. Z'_∞ is a f.g. torsion $\Lambda(\Gamma)$-module, with no non-zero finite $\Lambda(\Gamma)$-submodule

Theorem A is proven!

$$\begin{tikzcd} & \mathcal{L}'_\infty \arrow[dl, dash, "X"'] \arrow[ddl, dash, bend left, "\text{Galois}"] \\ F_\infty \arrow[d, dash, "\Gamma"'] & \\ F & \end{tikzcd}$$

$$(X)_\Gamma = \mathrm{Gal}(L'/F_\infty)$$

$$\begin{array}{ll} M_\infty & \\ \Big| & \text{No non-zero finite } \Lambda(\Gamma) \text{ submodule} \\ N_\infty & \\ \Big| & \mathrm{Gal}(N'_\infty/F_\infty) \text{ has no } \mathbb{Z}_p\text{-torsion} \\ F_\infty & \end{array}$$

$\Longrightarrow$ $\mathrm{Gal}(M_\infty/F_\infty)$ has no non-zero finite $\Lambda(\Gamma)$-submodule).

$$X(F_\infty)/\#(X(F_\infty)) \cdot [F:\mathbb{Q}]/2$$

$$0 \to X(F_\infty) \to \Lambda \to D \to 0$$

$$X(F_\infty) = \mathrm{Gal}(M_\infty/F_\infty)$$

D finite

$$\begin{tikzcd}
 & F_\infty \arrow[dl, dash] \arrow[d, dash] \\
F \arrow[d, dash] & K_\infty \arrow[dl, dash] \\
K &
\end{tikzcd}$$