$[F:\mathbb{Q}] < \infty$, $F_\infty = F(\mu_{p^\infty})$, $\Gamma = \mathrm{Gal}(F_\infty/F)$

E'_n = p-units of F_n, $\mathcal{E}'_n = E'_n/W_n$ $(0 \leq n \leq \infty)$

<u>Defn</u>. $\mathbb{Q}'$ = ring of rational numbers whose denominator is a power of p.

$$\mathbb{Q}'/\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p$$

We have exact sequence $(0 \leq n < \infty)$

$$0 \to \mathcal{E}'_n \to \mathcal{E}'_n \underset{\mathbb{Z}}{\otimes} \mathbb{Q}' \to \mathcal{E}'_n \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

Pass to inductive limit as $n \to \infty$

$$0 \to \mathcal{E}'_\infty \to \mathcal{E}'_\infty \underset{\mathbb{Z}}{\otimes} \mathbb{Q}' \to \mathcal{E}'_\infty \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

$\mathcal{E}_n'$ is a direct summand of $\mathcal{E}_\infty'$.

$$(\mathcal{E}_\infty')^{\Gamma_n} = \mathcal{E}_n'.$$

$$\Rightarrow \left(\mathcal{E}_\infty' \otimes_{\mathbb{Z}} \Phi'\right)^{\Gamma_n} = \mathcal{E}_n' \otimes_{\mathbb{Z}} \Phi'.$$

Also

$$H^1\left(\Gamma_n, \mathcal{E}_\infty' \otimes_{\mathbb{Z}} \Phi'\right) = \varinjlim_{m \geq n} H^1\left(\mathrm{Gal}(F_m/F_n), \mathcal{E}_m' \otimes \Phi'\right)$$

$H^1\left(\mathrm{Gal}(F_m/F_n), \mathcal{E}_m' \otimes \Phi'\right) = 0$ because

$\mathcal{E}_m' \otimes_{\mathbb{Z}} \Phi'$ is p-divisible.

$$\Rightarrow H^1\left(\Gamma_n, \mathcal{E}_\infty' \otimes_{\mathbb{Z}} \Phi'\right) = 0$$

Take Γ_n-cohomology of

$$0 \to \mathcal{E}'_\infty \to \mathcal{E}'_\infty \otimes \mathbb{Q}' \to \mathcal{E}'_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

<u>Proposition</u>. For all $n \geq 0$, we have the exact sequence

$$0 \to \mathcal{E}'_n \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p/\mathbb{Z}_p \to \left(\mathcal{E}'_\infty \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p/\mathbb{Z}_p\right)^{\Gamma_n} \to H^1(\Gamma_n, \mathcal{E}'_\infty) \to 0.$$

How big is $H^1(\Gamma_n, \mathcal{E}'_\infty)$?

<u>Proposition</u>. $A_n' = p$-primary subgroup of I_n'/P_n'. For all $n \geq 0$, we have

$$H^1(\Gamma_n, \mathcal{E}_\infty') \cong \mathrm{Ker}(A_n' \longrightarrow A_\infty').$$

In particular, $H^1(\Gamma_n, \mathcal{E}_\infty')$ is always a finite group.

Proof later in lecture

<u>Consequence</u>: s_n = number of primes of F_n above p

Dirichlet - Chevalley

$$\mathcal{E}_n' \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p/\mathbb{Z}_p = (\mathbb{Q}_p/\mathbb{Z}_p)^{r_2 p^n + s_n - 1}$$

Combined with exact sequence above gives: —

<u>Conclusion</u>. Maximal divisible subgroup of $(\mathcal{E}'_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n}$ is $(\mathbb{Q}_p/\mathbb{Z}_p)^{r_2 p^n + s_n - 1}$.

<u>Defn</u>. $\mathfrak{X}'_\infty = \mathrm{Hom}(\mathcal{E}'_\infty \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p)$

$\Longrightarrow (\mathfrak{X}'_\infty)_{\Gamma_n}$ is dual to $(\mathcal{E}'_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n}$ by Pontrjagin duality.

<u>Conclusion</u> $\mathbb{Z}_p$-rank of $(\mathfrak{X}'_\infty)_{\Gamma_n}$ is $r_2 p^n + s_n - 1$ for all $n \geqslant 0$.

$\mathfrak{X}'_\infty$ is a f.g. $\Lambda(\Gamma)$-module since $(\mathfrak{X}'_\infty)_\Gamma$ is a f.g. $\mathbb{Z}_p$-module.

<u>Fact</u>. There exists $n_0 \geqslant 0$, such that all primes of F_{n_0} above p are totally ramified in F_∞.

p totally ramified in $\mathbb{Q}(\mu_{p^\infty})$.

<u>Consequence</u>. $s_n = s_{n_0} = s$ for all $n \geqslant n_0$.

<u>Conclusion</u>. $(Y'_\infty)_{\Gamma_n}$ has $\mathbb{Z}_p$-rank $p^n \cdot r_2 + s - 1$ for all $n \geqslant n_0$.

Hence structure theory $\Longrightarrow$

<u>Theorem</u>. Y'_∞ has $\Lambda(\Gamma)$-rank equal to r_2.

$$\begin{array}{c} M_\infty \\ | \\ N'_\infty \\ | \\ F_\infty \end{array}$$

Kummer theory $\Rightarrow$

$$\mathrm{Gal}(N'_\infty/F_\infty) = \mathrm{Hom}(\mathcal{E}'_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})$$

with natural Γ-action

Hence:

$$\mathrm{Gal}(N'_\infty/F_\infty) = Y'_\infty \underset{\mathbb{Z}_p}{\otimes} T_p(\mu)$$

$$T_p(\mu) = \varprojlim \mu_{p^n}$$

Γ acts on $Y'_\infty \underset{\mathbb{Z}_p}{\otimes} T_p(\mu)$ by $\sigma(a \otimes b) = \sigma a \otimes \sigma b$.

<u>Fact</u>. W f.g. $\Lambda(\Gamma)$-module. $W \underset{\mathbb{Z}_p}{\otimes} T_p(\mu)$ has same $\Lambda(\Gamma)$-rank as W

Hence we have proven :-

<u>Theorem B</u>. $\mathrm{Gal}(N'_\infty/F_\infty)$ has $\Lambda(\Gamma)$-rank equal to $r_2 = [F:\mathbb{Q}]/2$.

How do we prove

$$H^1(\Gamma_n, \mathcal{E}'_\infty) = \mathrm{Ker}(A'_n \to A'_\infty) = H^1(\Gamma_n, E'_\infty)$$

Suffices to prove :-

<u>Proposition</u>. For all $m \geqslant n$, we have an isomorphism

$$\tau_{n,m} : \mathrm{Ker}(A'_n \to A'_m) \simeq H^1(\mathrm{Gal}(F_m/F_n), E'_m).$$

Fix a generator σ of $\mathrm{Gal}(F_m/F_n)$

$\mathcal{O}_m' =$ ring of p-integers of F_m

$c \in \mathrm{Ker}\,(A_n' \to A_m')$

Take $\mathfrak{a} \in I_n'$ in class of c

$$\mathfrak{a}\,\mathcal{O}_m' = \alpha\,\mathcal{O}_m' \qquad \alpha \in \mathcal{O}_m'$$

$$\varepsilon = \sigma\alpha/\alpha$$

<u>Observe</u>: $\varepsilon \in E_m'$, $N_{F_m/F_n}(\varepsilon) = 1$.

$\tau_{n,m}(c) =$ cohomology class of ε in $H^1(\mathrm{Gal}(F_m/F_n), E_m')$.

Well defined and a homomorphism

Also $\tau_{n,m}$ is injective

<u>Surjectivity</u>. Take any cohomology class in $H^1(\mathrm{Gal}(F_m/F_n), E'_m)$ represented by $\Theta \in E'_m$ with $N_{F_m/F_n}\Theta = 1$.

<u>Hilbert 90</u>: $\exists\, \alpha \in \mathcal{O}'_m$ with
$\Theta = \alpha^{\sigma - 1}$

Take $\mathfrak{a} \in I'_m$ defined by $\mathfrak{a} = \alpha\mathcal{O}'_m$.
$\mathfrak{a}^\sigma = \mathfrak{a}$ since $\alpha^{\sigma-1} = \Theta \in E'_m$.
But all primes of F_n which do not divide p are <u>unramified</u> in F_m/F_n
$\Rightarrow$ $\mathfrak{a}$ is the image of an ideal $\mathfrak{b}$ in I'_n.
Take c = class of $\mathfrak{b}$.
$\tau_{n,m}(c)$ = class of Θ

Greenberg's thesis gave examples where $\mathrm{Ker}(A_n' \longrightarrow A_\infty') \neq 0$.

Also Iwasawa proves:–

Theorem. $\mathrm{Gal}(N_\infty'/F_\infty)$ is a free $\mathbb{Z}_p$-module and, writing $t(\mathrm{Gal}(N_\infty'/F_\infty))$ for its $\Lambda(\Gamma)$-torsion submodule, then

$$\mathrm{Gal}(N_\infty'/F_\infty)/t(\mathrm{Gal}(N_\infty'/F_\infty))$$

is a free $\Lambda(\Gamma)$-module if and only if $H^1(\Gamma_n, E_\infty') = 0$ for all $n \geqslant n_0$.

Same for $\mathrm{Gal}(M_\infty/F_\infty)/t(\mathrm{Gal}(M_\infty/F_\infty))$.
Relevant for higher K-theory of $\mathcal{O}_F$.

M a f.g. torsion $\Lambda(\Gamma)$
- module

$\alpha(M)$ - adjoint of M.

(i) $\alpha(M)$ is pseudo-isomorphic to M.

$$\alpha(M) = \mathrm{Ext}^1_{\Lambda(\Gamma)}(M, \Lambda(\Gamma))$$

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