L 2

$[F:\mathbb{Q}]<\infty$, F_∞/F any $\mathbb{Z}_p$-extension, $\Gamma = \mathrm{Gal}(F_\infty/F)$.

Fact. If a place v of F ramifies in F_∞, then $v \mid p$.

Defn. M_∞ = max. abelian p-extension of F_∞ unramified outside p.

$X(F_\infty) = \mathrm{Gal}(M_\infty/F_\infty)$

$$
\begin{tikzcd}
 & M_\infty \arrow[dl, dash, "X(F_\infty)"'] \arrow[ddl, dash] \\
F_\infty \arrow[d, dash, "\Gamma"'] & \\
F &
\end{tikzcd}
$$

M_∞ is Galois over F by maximality

$$0 \longrightarrow X(F_\infty) \longrightarrow \mathrm{Gal}(M_\infty/F) \longrightarrow \Gamma \longrightarrow 0$$

$$\begin{array}{ccc} \mathrm{Gal}(M_\infty/F) & & \Gamma \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \tilde{\tau} & \longmapsto & \tau \end{array}$$

Γ acts on $X(F_\infty)$: $\tau x = \tilde{\tau} x \tilde{\tau}^{-1}$

<u>Question</u>. What is $X(F_\infty)_{\Gamma_n}$?

$$X(F_\infty)_{\Gamma_n} = \mathrm{Gal}(M_n/F_\infty).$$

<u>Here</u> M_n = max. abelian extension of F_n in M_∞.

<u>Lemma</u>. M_n = max. abelian p-extension of F_n unram. outside p.

Because only primes dividing p ramify in F_∞/F.

By global class field theory:

$\mathbb{Z}_p$-rank of $\mathrm{Gal}(M_n/F_n) = r_{2,n} + 1 + \delta_{F_n, p}$

where $r_{2,n}$ = number of complex primes of F_n.

But $r_{2,n} = r_2 p^n$ (even for $p = 2$).

Hence

$$(X(F_\infty))_{\Gamma_n} \text{ has } \mathbb{Z}_p\text{-rank } r_2 p^n + \delta_{F_n, p}$$

because $\mathrm{Gal}(F_\infty/F_n)$ has $\mathbb{Z}_p$-rank 1.

<u>Proposition</u>. $X(F_\infty)$ has $\Lambda(\Gamma)$-rank $\geqslant r_2$.

$X(F_\infty)$ has $\Lambda(\Gamma)$-rank equal to r_2

$\Longleftrightarrow$ $\delta_{F_n, p}$ are bounded as $n \to \infty$.

<u>Remark</u> $\delta_{F,p} = 0 \Rightarrow \delta_{F_n, p}$ are bounded as $n \to \infty$.

<u>Theorem</u> (Iwasawa). Assume F_∞/F is the <u>cyclotomic</u> $\mathbb{Z}_p$-extension. Then $\delta_{F_n,p}$ are bounded as $n \to \infty$.

<u>Idea</u> Use multiplicative Kummer theory.

<u>Assume</u>. $\mu_p \subset F$ if $p > 2$, $\mu_4 \subset F$ if $p = 2$.
Hence $F_\infty = F(\mu_{p^\infty})$.

<u>Kummer theory</u>. F_∞^{ab} = max. abelian p-extension of F_∞.

$$\langle\ ,\ \rangle : Gal(F_\infty^{ab}/F_\infty) \times (F_\infty^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p) \to \mu_{p^\infty}$$

$$\langle \sigma, \alpha \otimes p^{-a} \bmod \mathbb{Z}_p \rangle = \sigma\beta/\beta \qquad \beta^{p^a} = \alpha$$

$$Gal(F_\infty^{ab}/F_\infty) \cong Hom(F_\infty^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}).$$

Natural actions of Γ are preserved

$$\mathrm{Gal}(M_\infty/F_\infty) = \mathrm{Hom}(\mathfrak{M}_\infty, \mu_{p^\infty}).$$

What is $\mathfrak{M}_\infty \subset F_\infty^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p$.

__Fact__. Only finitely many primes of F_∞ above each prime of v, and these are discrete when $v \nmid p$.

I'_∞ = free abelian group on primes of F_∞ which do not lie above p.

$\alpha \in F_\infty^\times$ has $(\alpha)' \in I'_\infty$.

__Lemma__. $\mathfrak{M}_\infty$ consists of all $\alpha \otimes p^{-a} \bmod \mathbb{Z}_p$ in $F_\infty^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p$ with $(\alpha)' \in I_\infty^{p^a}$.

__Defn__. E'_∞ is set of all α in $F_\infty^\times$ with $(\alpha)' = 1$.

$$i_\infty : E'_\infty \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathfrak{M}_\infty \qquad (\alpha') = \varpi^{p^a}$$

$$j_\infty : \mathfrak{M}_\infty \longrightarrow A'_\infty$$

$A'_\infty = p$-primary sgp of I'_∞ / P'_∞ (P'_∞ = principal ideals)

$$j_\infty(\alpha \otimes p^{-a} \bmod \mathbb{Z}_p) = \text{class of } \varpi$$

__Lemma__. The sequence of Γ-modules

$$0 \to E'_\infty \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \to \mathfrak{M}_\infty \to A'_\infty \to 0$$

is exact.

We can use this to define N'_∞ by

$$N'_\infty = F_\infty\left(\sqrt[p^n]{\varepsilon} \ : \text{all } \varepsilon \text{ in } E'_\infty \text{ and all } n \geq 1\right).$$

$$\begin{array}{c} M_\infty \\ | \\ N'_\infty \\ | \\ F_\infty \end{array}$$

$$\begin{array}{c} M_\infty \\ | \\ N'_\infty \\ | \\ F_\infty \end{array} \qquad \begin{array}{l} \mathrm{Gal}(N'_\infty/F_\infty) = \mathrm{Hom}(E'_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}) \\[2em] \mathrm{Gal}(M_\infty/N'_\infty) = \mathrm{Hom}(A'_\infty, \mu_{p^\infty}). \end{array}$$

Proof now breaks up into two parts:-

<u>Theorem A</u>. For every $\mathbb{Z}_p$-extension F_∞/F $\mathrm{Hom}(A'_\infty, \mathbb{Q}_p/\mathbb{Z}_p)$ is a f.g. <u>torsion</u> $\Lambda(\Gamma)$-module.

<u>Note</u>: $A'_\infty = \varinjlim A'_n$ for an arbitrary $\mathbb{Z}_p$-extens

<u>Note</u> $\mathrm{Hom}(A'_\infty, \mathbb{Q}_p/\mathbb{Z}_p)$ torsion as a $\Lambda(\Gamma)$-module $\Rightarrow$ $\mathrm{Hom}(A'_\infty, \mathbb{Q}_p/\mathbb{Z}_p)(1) = \mathrm{Hom}(A'_\infty, \mu_{p^\infty})$ is $\Lambda(\Gamma)$-torsion when $F_\infty \supset \mu_{p^\infty}$

Theorem B (Iwasawa). Assume $\mu_{p^\infty} \subset F_\infty$. Then $\mathrm{Gal}(N'_\infty/F_\infty)$ is a f. g. $\Lambda(\Gamma)$-module of $\Lambda(\Gamma)$-rank $r_2 = [F:\mathbb{Q}]/2$.

In fact, we will see Iwasawa's proof even determines the $\Lambda(\Gamma)$-torsion submodule of $\mathrm{Gal}(N'_\infty/F_\infty)$ & more.

We need some elementary properties of E'_n & E'_∞.

W_n = group of roots of unity in F_n $(0 \leq n \leq \infty)$

__Defn__. $\mathcal{E}'_n = E'_n/W_n$, $\mathcal{E}'_\infty = E'_\infty/W_\infty$.

s_n = number of primes of F_n above p.

__Dirichlet-Chevalley__: $\mathcal{E}'_n$ is free of rank $r_2 p^n + s_n - 1$.

Obvious $\mathcal{E}_\infty'$ is a torsion free abelian group, but <u>not</u> obvious that

<u>Lemma</u>. $\mathcal{E}_\infty'$ is a free abelian group and $\mathcal{E}_n'$ is a direct summand for all $n \geqslant 0$.

<u>Proof</u>. $(E_\infty')^{\Gamma_n} = E_n'$, $H^1(\Gamma_n, W_\infty) = (W_\infty)_{\Gamma_n} = 0$.

$\Rightarrow (\mathcal{E}_\infty')^{\Gamma_n} = \mathcal{E}_n'$ for all $n \geqslant 0$.

$\mathcal{E}_\infty'$ is union of all $\mathcal{E}_n'$ $(n \geqslant 0)$.

<u>Key remark</u>. $\mathcal{E}_\infty' / \mathcal{E}_n'$ is torsion free.

$u \in \mathcal{E}_\infty'$ with $u^k \in \mathcal{E}_n'$

$\gamma \in \Gamma_n$ $(\gamma u / u)^k = 1 \Rightarrow \gamma u = u$

since $\mathcal{E}_\infty'$ is torsion free $\Rightarrow u \in \mathcal{E}_n'$.

$\mathcal{E}_m' / \mathcal{E}_n'$ torsion free for all $m \geqslant n$

$\mathcal{E}_n'$ is a direct summand of $\mathcal{E}_m'$.