$[F:\mathbb{Q}]<\infty$, F_∞/F, $\Gamma = \mathrm{Gal}(F_\infty/F)$

$\Gamma \xrightarrow{\sim} \mathbb{Z}_p \quad \Gamma_n \xrightarrow{\sim} p^n\mathbb{Z}_p$,

$F_\infty^{\Gamma_n} = F_n \quad \mathrm{Gal}(F_n/F) = \mathbb{Z}/p^n\mathbb{Z}$

$F = F_0 \subset F_1 \subset \dots F_n \subset \dots F_\infty = \bigcup F_n.$

<u>FACET 1</u>. F_∞/F

<u>Classical Iwasawa theory</u>.

p-adic behaviour of ideal class groups and units in F_∞/F, and interpretation via global class field theory.

$$\zeta(s) = \prod_p (1-p^{-s})^{-1}$$

(a). Complex zeroes of $\zeta(s)$
$\longleftrightarrow$ distribution of prime numbers.

(b) $\zeta(1-n) \in \mathbb{Q}$ $(n=2,4,6,\ldots)$

Kummer $\longleftrightarrow$ class number of $\mathbb{Q}(\mu_p)^+$.

Leopoldt-Kubota: p-adic analogue of $\zeta(s)$.

<u>Iwasawa</u>: zeroes of p-adic analogue of $\zeta(s)$
$\longleftrightarrow$ ~~p-ad~~ classical Iwasawa theory of
$$\mathbb{Q}(\mu_{p^\infty})^+ / \mathbb{Q}(\mu_p)^+$$

"Main Conjecture"
Mazur & Wiles complete proof.

$\zeta(s)$ has a simple zero at $s = -n$, $n = 2, 4, 6, \ldots$

$K_{2n}\mathbb{Z}$ $\quad n = 1, 2, \ldots$

<u>Borel - Garland</u> $K_{2n}\mathbb{Z}$ are finite groups.

Birch-Tate; Lichtenbaum

Theorem

$$\#\left(K_{2n-2}\mathbb{Z}\right) = \left|w_n(\mathbb{Q})\,\zeta(1-n)\right|$$

$$n = 2, 4, 6, \ldots$$

FACET 2.

$[F:\mathbb{Q}]$, M/F

Ex. $F=\mathbb{Q}$, $M =$ ell. curve $E/\mathbb{Q}$

$L(E,s)$ - entire

<u>BSD.</u> Precise relation between $E(\mathbb{Q})$ & $Ш(E/\mathbb{Q})$ & behaviour of $L(E,s)$ at $s=1$.

$\#(Ш(E/\mathbb{Q}))$ <u>exact formula</u>.

<u>Conjecture.</u>

$L(E,1) \neq 0 \Leftrightarrow E(\mathbb{Q})$ is finite & $Ш(E/\mathbb{Q})(p)$ is finite, p a prime.

'$\Longrightarrow$' Kolyvagin - Gross. Zag

'$\Longleftarrow$' Known for most E & $p >> 0$.

<u>Ex.</u> Prove '$\Longleftarrow$' for $p=2$ & $E: y^2 = x^3 - N^2 x$.

FACET 3. F_∞/F

FACET 4

Weber & Fukuda · Komatsu

Conj. $\mathbb{Q}_\infty/\mathbb{Q}$ any prime p.
$\mathbb{Q}_n$ - n-th layer
For every n & every p,
$\mathbb{Q}_n$ has class number 1.

<u>algebraic theory</u>.

$$\Lambda(\Gamma) = \varprojlim \mathbb{Z}_p[\Gamma/\Gamma_n]$$
$$\simeq \mathbb{Z}_p[[T]].$$

γ top. gen. of Γ

$\gamma \longmapsto 1+T$

<u>Fact</u> X - profinite abelian ~~p-adic~~ $\mathbb{Z}_p$-module.

X is f.g. over $\Lambda(\Gamma)$

$\Longleftrightarrow (X)_\Gamma = X/(\gamma-1)X$ f.g. over $\mathbb{Z}_p$.

$\mathcal{R}(\Gamma)$ - f.g. $\Lambda(\Gamma)$-modules.

Theorem. $M \in \mathcal{R}(\Gamma)$.

Then we have an exact sequence of $\Lambda(\Gamma)$-modules

$$0 \to D_1 \to M \longrightarrow \Lambda(\Gamma)^r \oplus \bigoplus_{i=1}^{k} \Lambda(\Gamma)/(f_i) \to D_2 \to 0$$

D_1, D_2 are finite $\Lambda(\Gamma)$-modules

Lemma: $\mathbb{Z}_p$-rank of $(M)_{\Gamma_n}$

$= p^n r + \delta$ for some fixed $\delta > 0$

when $n >> 0$.

Class field Theory

$[F:\mathbb{Q}] < \infty$, p any prime

M/F - max. abelian p-ext. of F in which only the primes $|p$ can ramify.

$Gal(M/F)$?

p-primary sgh of ideal class group of F

$$\begin{tikzcd}
 & Gal(M/F) \arrow[r] & G(L/F) \arrow[r] & 0 \\
Gal(M/L) \arrow[ur] & & & \\
0 \arrow[u] & & &
\end{tikzcd}$$

$$U_F = \prod_{v|p} U_{v,1}$$

E_F - global units $\equiv 1 \bmod v$ $\forall v/p$.

$i : E_F \hookrightarrow U_F$.

$\overline{E}_F$ – closure of E_F in p-adic topology

E_F has $\mathbb{Z}$-rank r_1+r_2-1.

$\overline{E}_F$ has $\mathbb{Z}_p$-rank $r_1+r_2-1-\delta_{F,p}$

$\delta_{F,p} \geqslant 0$.

defect of Leopoldt