The Fargues-Fontaine Curve

$C/\mathbb{F}_p$ perf. field, alg. closed $\varpi \in C$

$\mathcal{Y}_C = \mathrm{Spa}\, W(C^\circ) \setminus \{p[\varpi]=0\} \circlearrowleft \phi$

$\mathcal{X}_C = \mathcal{Y}_C/\phi^{\mathbb{Z}}$.

$B_C = H^0(\mathcal{Y}_C, \mathcal{O}_{\mathcal{Y}_C}) \circlearrowleft \phi$

$\mathcal{O}(1) =$ descent of $\mathcal{O}_{\mathcal{Y}_C} e$, $\phi(e) = p^{-1}e$

$H^0(\mathcal{X}_C, \mathcal{O}(1)) = (B_C e)^{\phi=1} = B_C^{\phi=p}$

$\simeq \tilde{H}(C^\circ)$, $H = \widehat{\mathbb{G}}_m$

Use $\mathcal{O}(1)$ to make "projective embedding"

let $P_d = H^0(\mathcal{X}_C, \mathcal{O}(d)) = B_C^{\phi=p^d}$

$P = \bigoplus_{d\geq 0} P_d$, $X_C = \mathrm{Proj}\, P$.

Thm (Fargues-Fontaine)

- $P_0 = \mathbb{Q}_p$
- P is graded factorial ring
- X_C is an int Noeth. dim 1 scheme
  if $x \in |X_C|$, $X_C \setminus \{x\} = \mathrm{Spec}(\mathrm{PID})$
- $\{\text{untilts of } C\} \xrightarrow{\sim} |X_C|$
  $\mathcal{O}_{X,x}/m_x \leftarrow x$

P analogous to $\mathbb{C}[x,y]$.
X analogous to $\mathbb{P}^1$

Hope: $\exists$ object $M/\mathbb{F}_p$ s.t. for S perfectoid char p,

$$M(S) = \{S \to M\} \simeq \{S^\#, S^{\#\flat} \xrightarrow{\sim} S, S^\# \to \mathrm{Spa}\,\mathbb{Q}_p\}$$

$$M = \mathrm{Spd}\,\mathbb{Q}_p \qquad (d = \text{diamond})$$

if $S = \mathrm{Spa}\,C$

$$M(S) = (\tilde{H}_{\mathbb{F}_p}(C^\circ)\setminus\{0\})/\mathbb{Z}_p^*$$

$$\tilde{H}_{\mathbb{F}_p} = \mathrm{Spa}\,\mathbb{F}_p[[T^{1/p^\infty}]] \circlearrowleft \underset{\substack{\Psi\\ a}}{\mathbb{Z}_p^*} : 1+T \mapsto (1+T)^a$$

$$= \mathrm{Spa}\,\mathbb{Q}_p^{cycl,\flat,\circ}$$

$$\tilde{H}_{\mathbb{F}_p}\setminus\{0\} = \mathrm{Spa}\,\mathbb{Q}_p^{cycl,\flat}$$

suggests

$$M = \text{“}\mathrm{Spa}\,\mathbb{Q}_p^{cycl,\flat}/\mathbb{Z}_p^*\text{”} =: \mathrm{Spd}\,\mathbb{Q}_p$$

$$(\mathbb{Q}_p^{cycl,\flat})^{\mathbb{Z}_p^*} = \mathbb{F}_p..$$

Say $K/\mathbb{F}_p$ perfectoid field

Given $K^\#/\mathbb{Q}_p$ untilt

Let $K^\#_\infty = \widehat{K^\#(\mu_{p^\infty})} \circlearrowleft$ perfectoid

$G = \mathrm{Gal}(K^\#(\mu_{p^\infty})/K^\#)$

Get $\mathbb{Q}_p^{cycl} \hookrightarrow K^\#_\infty$ $\quad K^\#_\infty/K^\#$ proétale

$\mathbb{Q}_p^{cycl,\flat} \hookrightarrow K^{\#\flat}_\infty = \underset{\substack{\circlearrowleft \\ G.}}{K_\infty}/K$ proétale

$\rightsquigarrow \quad \varepsilon \in \left[(\mathrm{Spa}\ \mathbb{Q}_p^{cycl,\flat})(K_\infty)/\mathbb{Z}_p^\times \right]^G$

$$\left\{ \begin{array}{l} K_\infty/K \text{ pro-étale, w/ group } G \\ \varepsilon \in \left[(\mathrm{Spa}\ \mathbb{Q}_p^{cycl,\flat})(K_\infty)/\mathbb{Z}_p^\times \right]^G \end{array} \right\} \to \left\{ \begin{array}{c} \text{untilts} \\ \text{of } K \end{array} \right\}$$

$$\begin{array}{c} \| \\ (K^\#, \iota) \\ \iota : K^{\#\flat} \xrightarrow{\sim} K \end{array}$$

Let Pfd = category of perf. spaces char p, w/ pro-étale topology.

Given $X \in \mathrm{Pfd}$, get $h_X : \mathrm{Pfd} \to \mathrm{Sets}$

$$Y \mapsto X(Y) = \mathrm{Hom}_{\mathrm{Pfd}}(Y, X)$$

Thm. h_X is a sheaf.

If X is a perf space, $X^\diamond = h_{X^\flat}$

If $X = \mathrm{Spa}\, R$, R perfectoid, $\mathrm{Spd}\, R = X^\diamond$.

$$\boxed{\begin{array}{c} \text{Perf spaces} \to \text{sheaves on Pfd} \\ X \mapsto X^\diamond \end{array}}$$

Def. $\mathrm{Spd}\, \mathbb{Q}_p = (\mathrm{Spd}\, \mathbb{Q}_p^{\mathrm{cycl}}) / \underline{\mathbb{Z}_p^\times}$ (as sheaf on Pfd)

Thus to give elt of $(\mathrm{Spd}\, \mathbb{Q}_p)(S)$ is to give:

- $\tilde{S} \to S$ a pro-étale cover
- $s \in \mathrm{Hom}(\tilde{S}, \mathrm{Spa}\, \mathbb{Q}_p^{\mathrm{cycl},\flat}) / \underbrace{\underline{\mathbb{Z}_p^*}(\tilde{S})}_{\mathrm{Hom}_{\mathrm{top}}(|\tilde{S}|, \mathbb{Z}_p^*)}$
- a descent datum for $\tilde{S} \to S$

Thm. $(\mathrm{Spd}\, \mathbb{Q}_p)(S) \cong \{ S^\# \to \mathrm{Spa}\, \mathbb{Q}_p, \; S^{\# \flat} = S \}$.

Def A diamond is a quotient of an object X of Pfd by a pro-étale equivalence relation

$$R \rightrightarrows X \qquad R(S) \rightrightarrows X(S) \rightsquigarrow \overset{\mathrm{im}}{R}(S) \subseteq X(S) \times X(S)$$

$$R^\diamond \rightrightarrows X^\diamond \to \mathcal{F}^\diamond$$

Perf. spaces : diamonds ::
schemes : alg. spaces.

Thm $\exists$ ~~fully faithful~~ functor

$$\{\text{analytic adic spaces}/\mathbb{Z}_p\ \cancel{(+\text{ normality cond})}\} \hookrightarrow \{\text{diamonds}\}$$

$$X \mapsto X^\diamond.$$

$$\underset{\substack{\uparrow \\ \mathrm{Pfd}}}{X^\diamond(Y)} = \{ Y^\#, \; Y^\# \to X, \; Y^{\# \flat} \cong Y \}.$$

Some $\mathbb{Q}_p$-vector space diamonds

Consider following morphisms

$$\mathrm{Pfd}_C \to \mathbb{Q}_p\text{-v.s.} \qquad C/\mathbb{Q}_p$$

- $\underline{\mathbb{Q}_p}$: sheafification of $R \mapsto \mathbb{Q}_p$.
- $\mathbb{A}_C^{1,\diamond}$: $R \mapsto R$
- $\tilde{H}_C^{\diamond}$: $R \mapsto \tilde{H}(R^\circ) = H^0(X_{R^\flat}, \mathcal{O}(1))$

$$0 \to \underline{\mathbb{Q}_p}(1) \to \tilde{H}_C^{\diamond} \to \mathbb{A}_C^{1,\diamond} \to 0$$

- For $\lambda \in \mathbb{Q}$ $0 \leq \lambda \leq 1$, $\lambda = d/h$

$$H_{\lambda} = p\text{-div gp}$$

$$\tilde{H}_{\lambda}(R^\circ) = H^0(X_{R^\flat}, \mathcal{O}(\lambda))$$

- Even for $\mathcal{O}(2)$

$$R \mapsto H^0(X_{R^\flat}, \mathcal{O}(2))$$

is also a diamond.

$$\left\{ \begin{array}{c} \text{perfectoid} \\ \text{spaces} / \mathbb{Q}_p \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{perf spaces} \\ X / \mathbb{F}_p \\ X \to \mathrm{Spd}\, \mathbb{Q}_p \end{array} \right\}.$$