①

The adic unit disc

$\mathrm{Spa}(\mathbb{C}_p\langle T\rangle, \mathbb{C}_p^\circ\langle T\rangle)$ - a connected <u>spectral</u> space containing $\mathbb{C}_p^\circ$

D

$\mathbb{C}_p^\circ$

Classification of points:

(1) Classical ($\mathbb{C}_p$-points)

(2) Gauß points (rat'l radius)

(3) Gauß points (irrat'l radius)

(4) Dead ends ($\mathbb{C}_p$ not sph.comp

(5) Rank 2 points.

$\overline{(2)} \supseteq (5)$.

Variations

$$\mathbb{A}^{1,ad}_{\mathbb{C}_p} = \varinjlim_{r} D$$

$$D \subseteq \mathbb{A}^{1,ad}_{\mathbb{C}_p} \quad \text{open}$$

$$\bar{D} = D \cup \{pt\} = \mathrm{Spa}(A, A^{++})$$

$$\mathbb{P}^{1,ad}_{\mathbb{C}_p} = \mathbb{A}^{1,ad}_{\mathbb{C}_p} \cup \{\infty\}$$

② The formal open disc

(p,T)-adic

$$X = \mathrm{Spa}(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]]) \to \mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$$
$$= \{s, \eta\}$$

$$X_{\mathbb{Q}_p} = X_\eta = \{x \in X \mid |p(x)| \neq 0\} \to \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) = \{\eta\}$$

If $x \in X_\eta$ $\exists\, n \gg 0$, $|T(x)^n| \leq |p|$

$$x \in \{x \in X \mid |T(x)|^n \leq |p| \neq 0\} = U\left(\frac{T^n}{p}\right)$$
$$= \mathrm{Spa}\Big(\underbrace{\mathbb{Z}_p[[T]]\left[\frac{T^n}{p}\right]^\wedge\left[\frac{1}{p}\right]}_{\left\{\sum_{n=0}^{\infty} a_n T^n \;\middle|\; a_n \in \mathbb{Q}_p \text{ conv. on } |T| \leq |p|^{1/n}\right\}}, \;\_\_\_\_\_\_ \Big)$$

$$X_\eta = \bigcup_{n=1}^{\infty} U\left(\frac{T^n}{p}\right)$$

not quasi-compact.

$$H^0(X_\eta, \mathcal{O}_{X_\eta}) = \left\{ \sum_{n=0}^{\infty} a_n T^n \;\middle|\; a_n \in \mathbb{Q}_p \text{ conv on } |T| < 1 \right\}.$$

not Huber ring

It's a Frechet algebra.

③ From last time:

$$H = \text{formal } \mathbb{G}_m$$

H: complete Huber pairs $/ (\mathbb{Z}_p, \mathbb{Z}_p) \longrightarrow \mathbb{Z}_p$-modules

$(R, R^+) \longmapsto 1 + R^{\circ\circ}$ under mult.

$R^{\circ\circ}$: top nilpotent elts

$$H = \mathrm{Spa}(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]])$$

$$H(R,R^+) = \mathrm{Hom}(\mathbb{Z}_p[[T]], R^+) = R^+ \cap R^{\circ\circ} = R^{\circ\circ}$$

$H_{\mathbb{Q}_p}$ is $\mathbb{Z}_p$-mod. object in category of adic spaces $/\mathbb{Q}_p$.

Let $\tilde{H} = \varprojlim_{p} H$ "universal cover"

$\mathbb{Q}_p$-vector space

$$\tilde{H}(R,R^+) = \varprojlim_{x \mapsto x^p} 1 + R^{\circ\circ}$$

$$\xrightarrow{\sim} \varprojlim_{x \mapsto x^p} 1 + R^{\circ\circ}/p \simeq \varprojlim R^{\circ\circ}/p \simeq \varprojlim_{x \mapsto x^p} R^{\circ\circ}$$

(p,T)-adically complete

$$\tilde{H} = \mathrm{Spa}(\mathbb{Z}_p[[T^{1/p^\infty}]], \mathbb{Z}_p[[T^{1/p^\infty}]])$$

For a perfectoid field K, $\tilde{H}_K$ is a perfectoid space

Last time: $\tilde{H}(C^\circ) = H(C^\circ) = 1 + m_C$

$C/\mathbb{F}_p$ alg. closed perf. field.

$\tilde{H}(C^\circ)/\mathbb{Z}_p^* \simeq \{\text{untilts of } C\}$

If $C^\#$ is an untilt: $\tilde{H}(C^{\#\circ}) \simeq \tilde{H}(C^\circ)$

$$0 \to \mu_{p^\infty}(C^\#) \to \tilde{H}(C^{\#\circ}) \xrightarrow{\log} C^\# \to 0$$

exact seq of $\mathbb{Z}_p$-modules:

apply $\varprojlim_p$

$$0 \to \mathbb{Q}_p(1) \to \underset{\tilde{H}(C^\circ)}{\underset{\parallel}{\tilde{H}(C^{\#\circ})}} \to C^\# \to 0$$

" of $\mathbb{Q}_p$-vector spaces

Colmez: Banach Space of dimension $(1, 1)$

Fargues-Fontaine curve.

$C/\mathbb{F}_p$ alg. closed perfectoid field $C \ni \varpi$

untilts of C of char 0 $\longleftrightarrow$ ~~etc~~ ideals $(\xi) \subseteq W(C^\circ)$

ξ is primitive deg 1.

For $C^\#$ untilt of C

$$\theta_{C^\#} : W(C^\circ)\left[\tfrac{1}{p[\varpi]}\right] \to C^\#$$

$$[x] \longmapsto x^\#$$

<u>Def</u> (adic FF curve)

$$\phi \curvearrowright \mathcal{Y}_C = \operatorname{Spa}(W(C^\circ), W(C^\circ)) \setminus \{x \mid |p[\varpi]|(x) = 0\}$$

$$\{p[\varpi]=0\}$$

$$\{\text{untilts of } C^\# \text{ to char } 0\} \to \mathcal{Y}_C.$$

<u>Thm</u> (Kedlaya) $\mathcal{Y}_C$ is an adic space

$$\phi \curvearrowright B_C := H^0(\mathcal{Y}_C, \mathcal{O}_{\mathcal{Y}_C}) \quad \text{Fréchet algebra.}$$

Given

$$\dot{\varepsilon} \in \tilde{H}(C^\circ) \simeq 1 + m_C \text{ , get}$$

$$[\varepsilon] \in H(W(C^\circ))$$

$$[\varepsilon]^\phi = [\varepsilon^p] = [\varepsilon]^p$$

$$t = \log[\varepsilon] = \sum (-1)^{n-1} \frac{([\varepsilon]-1)^n}{n} \in B_C$$

$(\approx \xi)$

$$t^\phi = \log [\varepsilon]^\phi = \log [\varepsilon]^p$$
$$= p\,t.$$

$$t \in B_C^{\phi = p}. \qquad x \in C^{\circ\circ}$$

$$\cup\!\!\!\!\!\!\!\cup$$

$$\cdots + p^2[x^{1/p^2}] + p[x^{1/p}] + [x] + \frac{[x^p]}{p} + \frac{[x^{p^2}]}{p^2} + \cdots$$

<u>Thm</u>. (Fargues-Fontaine)

$$\tilde{H}(C^\circ) \xrightarrow{\sim} B_C^{\phi=p}$$

If $C^\#$ is an untilt to char 0,
$\theta_{C^\#} : W(C^\circ) \to C^\#$ extends to B_C

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & \tilde{H}(C^\circ) & \xrightarrow{\log} & C^\# & \longrightarrow & 0 \\
 & & \downarrow \simeq & & \wr\| & & \| & & \\
0 & \longrightarrow & \mathbb{Q}_p t & \longrightarrow & B_C^{\phi=p} & \xrightarrow[\theta_{C^\#}]{} & C^\# & \longrightarrow & 0
\end{array}$$

Variation $\quad d, h \geq 1$ rel prime $\quad d \leq h$

$H_{d/h}$ formal group ht h dim $d / \mathbb{Z}_p$

eg E ss. $\rightsquigarrow$ $\hat{E}_0 \simeq H_{1/2}$.

$$H_{d/h} \simeq \mathrm{Spf}\, \mathbb{Z}_p [\![T_1, \dots, T_d]\!]$$

$$\tilde{H}_{d/h} \simeq \mathrm{Spf}\, \mathbb{Z}_p [\![T_1^{1/p^\infty}, \dots, T_d^{1/p^\infty}]\!]$$

$\tilde{H}_{d/h, K}$ is a $\mathbb{Q}_p$-v.s. object in cat. of perf spaces.

$$0 \to \mathbb{Q}_p^h \to \tilde{H}_{d/h}(C^\circ) \to (C^\#)^d \to 0$$

$$\begin{array}{c} \tilde{H}_{d/h}(C^\circ) \\ \wr\wr \\ B_C^{\phi^h = p^d} \end{array}$$