Intro to adic spaces

$$\begin{array}{ccc} & & \text{Perf Spaces} \\ & & \downarrow \\ \begin{array}{l} * \text{ Schemes} \\ \text{Formal Sch} \\ \text{Rigid Spaces} \\ \hline \text{Berk Spaces} \end{array} & \hookrightarrow & \text{Adic Spaces.} \\ & & \downarrow \\ \begin{array}{l} \text{manifolds} \\ \text{cpx analytic spaces} \end{array} & \hookrightarrow & \begin{array}{c} \text{Ringed Spaces} \\ (X, \mathcal{O}_X) \end{array} \end{array}$$

Rigid Spaces (eg rigid closed unit disc / $\mathbb{Q}_p$)

$$A = \mathbb{Q}_p\langle T\rangle = \left\{ a_0 + a_1 T + a_2 T^2 + \dots \;\middle|\; \begin{array}{l} a_i \in \mathbb{Q}_p \\ |a_i| \to 0 \end{array} \right\}$$

(converge on $|T| \leq 1$)

$$X = \operatorname{Spm} \mathbb{Q}_p\langle T\rangle \simeq \{x \in \overline{\mathbb{Q}}_p, \ |x| \leq 1\} / G_{\mathbb{Q}_p}$$

$(f(x))$

$f \in \mathbb{Q}_p[T]$

want topology on X st (eg $H^0(X, \mathcal{O}_X) = A$)

Problem: "native top." is tot. disconnected

Solution (Tate): Put a G-topology on X, with "adm. opens" and "adm. covers":

Eg. $Y := \{|T|<1\} \cup \{|T|=1\} = X$

is not an admissible cover of X.

$Y \to X$ is a bij. on points but not an isom

Other problem: scope is too narrow

Eg No "$\mathrm{Spm}\, \mathbb{Z}_p\langle T\rangle \to \mathrm{Spec}\, \mathbb{Z}_p$"

w/ generic fiber $\mathrm{Spm}\, \mathbb{Q}_p\langle T\rangle$.

③ Huber rings

<u>Def</u>. A top. ring A is <u>Huber</u> (f-adic) if $\exists$ open subring $A_0 \subseteq A$ (ring of definition) whose topology is I-adic for a finitely generated ideal $I \subseteq A_0$.

<u>Eg</u>. Any ring R w/ disc. top. $A_0 = R$, $I = 0$

- $\mathbb{Q}_p$ $\quad A_0 = \mathbb{Z}_p$, $\quad I = (p)$, or $(p^2)\ldots$
- $\mathbb{Q}_p\langle T\rangle$ $\quad A_0 = \mathbb{Z}_p\langle T\rangle$, $\quad I = (p)$
$$= \mathbb{Z}_p[T]^\wedge_{(p)}$$
- $R[[T_1,\ldots,T_n]] = A$, $\quad A_0 = A$, $\quad I = (T_1,\ldots,T_n)$
- $\mathbb{Z}_p[[T_1,\ldots,T_n]] = A$, $\quad A_0 = A$, $\quad I = (p, T_1,\ldots,T_n)$
- K nonarch. field perfect of char p

$W(\mathcal{O}_K) = A$, $\quad A_0 = A$, $\quad I = (p, [\varpi])$.

$\varpi \in \mathcal{O}_K$, $\quad 0 < |\varpi| < 1$

Given a Huber ring A, let $A^\circ \subseteq A$ be subring of power-bounded elements

$(\mathbb{Q}_p\langle T\rangle^\circ = \mathbb{Z}_p\langle T\rangle.)$

A is <u>uniform</u> if A° is bounded

<u>Eg</u> $A = \mathbb{Q}_p[T]/T^2$

$A^\circ = \mathbb{Z}_p \oplus \mathbb{Q}_p T$, A not uniform

A is <u>Tate</u> if it contains a <u>topologically nilpotent</u> unit.
(a pseudo-uniformizer)

Uniform Tate rings A are nice:
$A_0 \subseteq A^\circ \subseteq A$ is a ring of def'n, w/ ideal (ϖ), $\varpi \in A$ pseudo-uniformizer.

$\mathbb{Q}_p$, $\mathbb{Q}_p\langle T\rangle$, not $\mathbb{Z}_p$, $R[[T_1, \dots, T_n]]$.

Such an A is Banach:
$a \in A$, $|a| = 2^{\inf\{n \,:\, \varpi^n a \in A^\circ\}}$

⑤ # Continuous Valuations

Modeled on $|\cdot| : \mathbb{Q}_p \to \mathbb{R}_{\geq 0}$

For a Huber ring A, a <u>cts. valuation</u> is

a $\quad |\cdot| : A \longrightarrow \Gamma \cup \{0\}$

$\uparrow$ tot. ordered ab. grp.
$(\mathbb{R}_{>0},\ \mathbb{R}_{>0} \times \mathbb{R}_{>0}, \dots)$

- $|ab| = |a|\,|b|$
- $|a+b| \leq \max(|a|, |b|)$
- $|1| = 1$
- $|0| = 0$
- $\forall \gamma \in \Gamma,\ \{a \in A \mid |a| < \gamma\}$ is open in A.

$|\cdot|$ and $|\cdot|'$ are <u>equivalent</u> if

$\forall a, b \in A, \qquad |a| \leq |b| \iff |a|' \leq |b|'$.

$x \in \mathrm{Cont}(A) = \{$eq. classes of cts valuations on $A\}$

top. is generated by $\{\underset{x \in \mathrm{Cont}(A)}{|f(x)| \leq |g(x)| \neq 0}\}$.

$|f(x)| = |f|_x$

(6)

$\mathrm{Cont}\ \mathbb{F}_p = \{|\cdot|_0\}$

$\mathrm{Cont}\ \mathbb{Q}_p = \{|\cdot|_p\}$

$\mathrm{Cont}\ \mathbb{Z}_p = \{s, \eta\}$ — $\mathbb{Z}_p \to \mathbb{Q}_p \xrightarrow{|\cdot|_p} \mathbb{R}_{\geq 0}$

$\mathbb{Z}_p \to \mathbb{F}_p \xrightarrow[|\cdot|_0]{} \{0,1\}$

$\{\eta\} = \{|p(x)| \neq 0\}$, $\overline{\{\eta\}} = \mathrm{Cont}\ \mathbb{Z}_p$

$\mathrm{Cont}\ \mathbb{Q}_p\langle T\rangle = ?$

contains more points.

$\mathrm{Spm}\ \mathbb{Q}_p\langle T\rangle \to \mathrm{Cont}\ \mathbb{Q}_p\langle T\rangle \ni x^-$

$M \longmapsto (\mathbb{Q}_p\langle T\rangle \to K \xrightarrow{|\cdot|_K} \mathbb{R}_{\geq 0})$

$K = \mathbb{Q}_p\langle T\rangle / M$

let $\Gamma = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$ $\quad a < \gamma < 1$

$\forall a \in \mathbb{R},\ a < 1$

$$\left|\sum a_n T^n\right|_{x^-} = \sup |a_n| \gamma^n$$

$$|T|_{x^-} = \gamma$$

$\bigcup_{n \geq 1} \{|T^n(x)| \leq |p|\}$ and $\{|T(x)| = 1\}$ do not cover $\mathrm{Cont}\ \mathbb{Q}_p\langle T\rangle$, b/c of x^-.

(∗) Cont $\mathbb{Q}_p\langle T\rangle$ also contains x^+, where $\gamma > 1$.

$A^+ \subseteq A^\circ$ ring of integral elements
open, integrally closed

$$\mathrm{Spa}(A, A^+) = \{x \in \mathrm{Cont}(A) \mid |f(x)| \le 1 \ \forall f \in A^+\}.$$

Eg $A = \mathbb{Q}_p\langle T\rangle$

$A^+ = \mathbb{Z}_p\langle T\rangle = A^\circ$, $\mathrm{Spa}(A, A^+) \not\ni x^+$.

$\cup$

$A^{++} = \{a_0 + a_1 T + \dots \mid a_0 \in \mathbb{Z}_p,\ a_i \in p\mathbb{Z}_p,\ i > 0\}$

$\mathrm{Spa}(A, A^{++}) = \mathrm{Cont}(A)$

$X = \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle) =$ adic closed unit disc.

$\mathcal{O}_X$ structure presheaf.